Total Points: 60

PSTAT 120B / **FINAL EXAMINATION** / Summer 2024 Instructor: **Ethan P. Marzban**

Name: ______________________________ **NetID:** ____________________
(*First and Last*) (*NOT Perm Number*)

Your Section: **2pm (Hyuk-Jean)** **3pm (Hyuk-Jean)** **4pm (Minwoo)** **5pm (Minwoo)**
(*Circle One*)

Instructions:

- You will have **2 hours** to complete this exam.
  - Nobody will be permitted to leave the exam room during the last 10 minutes of the exam.
- Please remember to write your name and NetID (not perm number) **at the top of each sheet** of this exam.
- You are allowed the use of **two 8.5 $\times$ 11-inch** sheets, front and back, of notes, along with the use of a **calculator**; the use of any and all other resources (including, but not limited to laptops, cell phones, textbooks, etc.) is prohibited.
- Unless otherwise specified, all multiple choice questions have **only one** correct answer.
  - Partial credit is **not** available on multiple choice questions.
- Unless otherwise specified, simplification is not needed; however, all integrals and infinite sums (unless otherwise specified) must be evaluated.
- **Good Luck!!!**

Honor Code: In signing my name below, I certify that all work appearing on this exam is entirely my own and not copied from any external source. I further certify that I have not received any unauthorized aid while taking this exam.

$\times$ ______________________________

1 Multiple Choice Questions

Please fill in the bubble(s) **on the exam below** *corresponding to your answer. You do not need to submit any additional work for these questions. No partial credit is available for multiple choice questions.*

1. (1 point) **True or False:** Given a sample $\vec{\boldsymbol{Y}} = \{Y_i\}_{i=1}^n$ from a distribution with unknown parameter θ, the entire sample $\vec{\boldsymbol{Y}}$ is a sufficient statistic for θ.

   ◯ True ◯ False

2. (1 point) Consider $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, 1)$, where $\mu \in \mathbb{R}$ is unknown. Is it possible to find an estimator $\widehat{\mu}_n$ for μ whose variance is smaller than the Cramér-Rao Lower Bound? (Read the problem statement carefully!)

   ◯ Yes, it is possible ◯ No, it is not possible

3. (1 point) Consider an i.i.d. sample $\vec{\boldsymbol{Y}} := \{Y_i\}_{i=1}^n$ from a distribution with unknown parameter θ. Suppose that $U := \sum_{i=1}^n \sqrt{Y_i}$ is a sufficient statistic for θ. Which of the following must also be a sufficient statistic for θ? (Remember, there is only one correct answer.)

   ◯ $\sum_{i=1}^n Y_i$
   ◯ $2\sum_{i=1}^n \sqrt{Y_i}$
   ◯ $\frac{1}{n}\sum_{i=1}^n Y_i$
   ◯ Y_1
   ◯ None of the above

4. (1 point) Which of the following properties does the maximum likelihood estimator **not** possess in general?

   ◯ Asymptotic normality
   ◯ Asymptotic efficiency
   ◯ Consistency
   ◯ Unbiasedness (for any sample size)
   ◯ None of the above

5. (1 point) If $(Y \mid X = x) \sim \text{Exp}(x)$ and $X \sim \text{Gamma}(3, 2)$, what is the correct value of $\mathbb{E}[Y]$?

   ◯ 1.500
   ◯ 6.000
   ◯ x [this is a lowercase x]
   ◯ X [this is an uppercase X]
   ◯ None of the above

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2 Free-Response Questions

1. (6 points) Let $Y_1, Y_2 \overset{\text{i.i.d.}}{\sim} \text{Exp}(\theta)$, and define $U := Y_2/Y_1$. Show that $F_U(u)$, the CDF (cumulative distribution function) of U, is given by

$$F_U(u) = \begin{cases} 0 & \text{if } u < 0 \\ \frac{u}{u+1} & \text{if } u \geq 0 \end{cases}$$

For full credit, you must set up and evaluate a double integral corresponding to $F_U(u)$, and you **must sketch the region of integration**.

2. Let $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim} f(y;\theta)$ where

$$f(y;\theta) = \theta y^{\theta-1} \cdot \mathbb{1}_{\{0<y<1\}}$$

and $\theta > 0$ us an unknown parameter.

(a) (3 points) Show that $U := \left(\prod_{i=1}^{n} Y_i \right)$ is a sufficient statistic for θ.

(b) (3 points) Let $W_i := -\ln(Y_i)$. Use any of the methods discussed in this course to show that $W_i \sim \text{Exp}(1/\theta)$.

(c) (3 points) Show that

$$V := \left(2\theta \sum_{i=1}^{n} [-\ln(Y_i)]\right) \sim \chi^2_{2n}$$

You may use any result from class, however you must clearly state which result/s you are using. Don't skip any steps!

3. Let $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim} f(y; \beta)$ where

$$f(y; \beta) = \frac{3\beta^3}{y^4} \cdot \mathbb{1}_{\{y \geq \beta\}}$$

and $\beta > 0$ is an unknown constant. A fact you may use without proof is that $\mathbb{E}[Y_1] = 3\beta/2$.

(a) (2 points) Derive an expression for $\widehat{\beta}_{\text{MM}}$, the method of moments estimator for β.

(b) (2 points) Show that $\widehat{\beta}_{\text{MM}}$ is a consistent estimator for β - don't just cite the property that method of moments estimators are typically consistent!

(c) (3 points) Derive an expression for $\mathcal{L}_{\vec{\boldsymbol{Y}}}(\beta)$, the likelihood of the sample $\vec{\boldsymbol{Y}} = \{Y_i\}_{i=1}^{n}$.

(d) (2 points) Use your answer to part (c) to find $\widehat{\beta}_{\text{MLE}}$, the method of moments estimator for β.

4. Let $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim} f(y;\theta)$ where

$$f(y;\theta) = \frac{3y^2}{\theta^3} \cdot \mathbb{1}_{\{0 \leq y \leq \theta\}}$$

and $\theta > 0$ is an unknown parameter. A fact you may use, without proof, is that the CDF (cumulative distribution function) of Y_1 is given by

$$F_{Y_1}(y) = \begin{cases} 0 & \text{if } y < 0 \\ (y/\theta)^3 & \text{if } 0 \leq y < \theta \\ 1 & \text{if } y \geq \theta \end{cases}$$

Additionally, define $U := Y_{(n)}/\theta$, where $Y_{(n)} := \max_{1 \leq i \leq n}\{Y_i\}$ denotes the sample maximum (i.e. the n^{th} order statistic).

(a) (3 points) Show that the density of $Y_{(n)}$, the n^{th} order statistic, is given by

$$f_{Y_{(n)}}(y) = \frac{3ny^{3n-1}}{\theta^{3n}} \cdot \mathbb{1}_{\{0 \leq y \leq \theta\}}$$

(b) (3 points) Show that U has density

$$f_U(u) = 3nu^{3n-1} \cdot \mathbb{1}_{\{0 \leq u \leq 1\}}$$

and use the definition of a pivotal quantity to argue that U is a pivotal quantity for θ.

(c) (4 points) Find constants a and b such that $\mathbb{P}(U < a) = \alpha/2$ and $\mathbb{P}(U > b) = \alpha/2$.

(d) (4 points) Combine your answers to parts (a) and (b) to construct a $(1 - \alpha) \times 100$% confidence interval for θ.

5. As she is watching the 2024 Paris Olympics, Biyonka becomes interested in performing inference on the true average length (in minutes) of a tennis match. Among an i.i.d. sample of 25 tennis matches, she observes a sample average length of 140 minutes and a sample standard deviation of 10 minutes. Suppose she wishes to test the claim that the true average length of a tennis match is 150 minutes against a two-sided alternative, using a 5% level of significance. Assume the lengths of tennis matches are normally distributed.

   (a) (1 point) Let μ denote the true average length (in minutes) of a tennis match. State the null and alternative hypotheses.

   (b) (1 point) Should Biyonka perform a $Z-$test or a $T-$test? Justify your answer.

   (c) (2 points) Compute the observed value of the test statistic.

(d) (2 points) Derive an expression for the $p-$value of the observed value of the test statistic. You may leave your answer in terms of the CDF of a distribution.

(e) (2 points) Suppose that the $p-$value associated with Biyonka's data is 0.01 (which is **<u>not</u>** to say this is the right answer to part (c) above!) State the conclusions of the test, and phrase your findings in the context of the problem.

6. Let $(Y_1, Y_2) \sim f_{Y_1,Y_2}(y_1, y_2) = 2 \cdot \mathbb{1}_{\{0 \leq y_1 < y_2 \leq 1\}}$.

   (a) (3 points) Derive an expression for $f_{Y_1}(y_1)$, the marginal density of Y_1. Be sure to include the support of Y_1 as well!

(b) (3 points) Derive an expression for $f_{Y_1|Y_2}(y_1 \mid y_2)$, the conditional density of $(Y_1 \mid Y_2 = y_2)$. Be sure to not only include the support, but also the values of y_2 for which the density is defined! A fact you may use without proof is that the marginal density of Y_2 is given by

$$f_{Y_2}(y_2) = 2y_2 \cdot \mathbb{1}_{\{0 \leq y_2 \leq 1\}}$$

(c) (3 points) Compute $\mathbb{E}[Y_1 \mid Y_2 = y_2]$.

You may use the remainder of this page for scratch work; please note that nothing written on this page will be graded.

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