Total Points: 25

PSTAT 120B / **MIDTERM 1** / Summer 2024 Instructor: **Ethan P. Marzban**

Name: ______________________________ **NetID:** ______________________
(*First and Last*) (*NOT Perm Number*)

Your Section: **2pm (Hyuk-Jean)** **3pm (Hyuk-Jean)** **4pm (Minwoo)** **5pm (Minwoo)**
(*Circle One*)

Instructions:

- You will have **55 minutes** to complete this exam.
  - Nobody will be permitted to leave the exam room during the last 10 minutes of the exam.
- Please remember to write your name and NetID (not perm number) at the top of each sheet of this exam.
- You are allowed the use of a single **8.5 $\times$ 11-inch** sheet, front and back, of handwritten notes. You are also permitted the use of **calculators**; the use of any and all other electronic devices (laptops, cell phones, etc.) is prohibited.
  - You will be asked to turn in your note sheet with your exam.
- Unless otherwise specified, simplification is not needed; however, all integrals and infinite sums (unless otherwise specified) must be evaluated.
  - One exception is that, whenever applicable, answers may be left in terms of Φ, the standard normal c.d.f..
- **Good Luck!!!**

Honor Code: In signing my name below, I certify that all work appearing on this exam is entirely my own and not copied from any external source. I further certify that I have not received any unauthorized aid while taking this exam.

$\times$ ____

1. Tasha and Tony have both gotten in line at the *Gaucho Boutique*, and are waiting to be served. Let Y_1 denote the amount of time (in minutes) Tasha waits in line, and let Y_2 denote the amount of time (in minutes) Tony waits in line. It is found that the joint density of (Y_1, Y_2) is given by

$$f_{Y_1,Y_2}(y_1, y_2) = e^{-y_1} \cdot \mathbb{1}_{\{0 \leq y_2 \leq y_1 < \infty\}}$$

(a) (4 points) Find $f_{Y_2}(y_2)$, the marginal density of Y_2, and use this to identify the distribution of Y_2 by name. Be sure to also include any/all relevant parameter(s)!

(b) (6 points) Given that Tasha ends up waiting for *exactly* 3 minutes, what is the probability that Tony ends up waiting for more than 2 minutes? You may use, without proof, the fact that $Y_1 \sim$ Gamma$(2, 1)$.

Name:__________________________________ **NetID:**____________________

2. A particular pesticide is composed of two compounds, called Compound A and Compound B. Suppose that the proportions Y_1 and Y_2 of compounds A and B, respectively, in a particular random sample of pesticide is given by

$$2 \cdot \mathbb{1}_{\{0 \leq y_1 \leq 1 \ , \ 0 \leq y_2 \leq 1 \ , \ 0 \leq y_1 + y_2 \leq 1\}}$$

(a) (1 point) One can show (and you do <u>not</u> need to prove this) that

$$f_{Y_1 | Y_2}(y_1 \mid y_2) = \frac{1}{1 - y_2} \cdot \mathbb{1}_{\{0 \leq y_1 \leq 1 - y_2\}}$$

Crucially, though, this density is missing a specification of what values of y_2 it is defined over. For what values of y_2 is the conditional density defined? Justify your answer.

(b) (2 points) Given that a sample of pesticide contains 70% Compound B, what is the expected percentage of Compound A contained in the sample? (You may still use the conditional density provided in the statement of part (a) without proof.)

(c) (4 points) Use the Law of Iterated Expectations to compute $\mathbb{E}[Y_1]$. You may use (without proof) the fact that $\mathbb{E}[Y_2] = 1/3$, along with the conditional density provided in part (a). Please note that if you simply double-integrate the joint density, you will not receive full points.

3. A random variable Y is said to follow the **Pareto Distribution**, notated $Y \sim$ Pareto(θ, α) for parameters $\theta > 0$ and $\alpha > 0$, if Y has density given by

$$f_Y(y) = \frac{\alpha \theta^\alpha}{y^{\alpha+1}} \cdot \mathbb{1}_{\{y \geq \theta\}}$$

Let $Y \sim$ Pareto(θ, α).

(a) (3 points) Define $U_1 := cY$ for a positive constant c. Derive the density $f_{U_1}(u)$ of U_1 using the Change of Variable Formula (aka the method of transformations). Be sure to include the support of U_1 as well.

(b) (1 point) Does U_1 (as defined in part (a) above) follow the Pareto distribution? If so, identify the parameters.

(c) (4 points) Define $U_2 := \sqrt{Y}$. Derive the density $f_{U_2}(u)$ of U_2 using any of the methods discussed in lecture

You may use the remainder of this page for scratch work; please note that nothing written on this page will be graded.