HOMEWORK 06

PSTAT 120B: Mathematical Statistics, I
Summer Session A, 2024 with Instructor: Ethan P. Marzban

1. In this problem, we'll re-do the exercise we started Thursday's (July 25) lecture with, just with a bit more guidance and a more "practical" final answer. Let $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim}$ Exp(θ); we ultimately wish to construct a $(1-\alpha) \times 100$% confidence interval for $\tau(\theta) := \theta^2$ (the population variance).

   (a) Find $\widehat{\tau}_{\text{MLE}}$, the maximum likelihood estimator for τ. You may use, without proof, the fact that the MLE for θ is given by $\overline{Y}_n$.

   (b) Identify the asymptotic sampling distribution of $\widehat{\tau}_{\text{MLE}}$, by appealing to the theorem titled "Theorem (Asymptotic MLE Result)" from lecture.

   (c) Use the asymptotic normality of the maximum likelihood estimator to show that a theoretical large-sample $(1-\alpha) \times 100$% confidence interval for τ is given by

   $$\overline{Y}_n^2 \pm \Phi^{-1}\left(1 - \frac{\alpha}{2}\right) \cdot \frac{2\theta^2}{\sqrt{n}}$$

   (d) Now, in practice, the interval posited in part (c) isn't really usable, because it depends on θ (whose value is unknown). As such, a more "practical" CI is obtained by replacing θ^2 with its MLE: $(\overline{Y}_n)^2$:

   $$\overline{Y}_n^2 \pm \Phi^{-1}\left(1 - \frac{\alpha}{2}\right) \cdot \frac{2\overline{Y}_n^2}{\sqrt{n}}$$

   (Though there is perhaps some question about whether or not replacing θ^2 with an estimator for θ^2 breaks the normality of our estimator, we won't concern ourselves with that for this course.)

   Suppose we obtain an observed sample of 100 values from the Exp(θ) distribution, and it is calculated that $\overline{y}_{100} = 2.720$. Construct a 95% CI for θ^2. Use a computer software (e.g. WolframAlpha, etc.) to compute any inverse-CDF values, so that you report only a numerical interval.

2. Consider $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$ where $\mu \in \mathbb{R}$ is unknown but $\sigma^2 > 0$ is known. In this problem, we will work toward constructing a lower-tailed test at an α level of significance.

   (a) Let $H_0 : \mu = \mu_0$. Write down the correct alternative hypothesis for our test (remember that we're trying to construct a lower-tailed test).

   (b) Should the rejection region be: "values of $\overline{Y}_n$ that are much larger than μ_0" or "values of $\overline{Y}_n$ that are much smaller than μ_0"? Justify your answer verbally.

   (c) Show that the critical value for the test must be $c := \Phi^{-1}(\alpha)$, if our test statistic is $\frac{\overline{Y}_n - \mu_0}{\sigma/\sqrt{n}}$