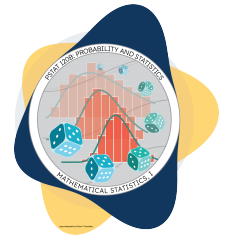


HOMWORK 06

PSTAT 120B: Mathematical Statistics, I
Summer Session A, 2024 with Instructor: Ethan P. Marzban



1. In this problem, we'll re-do the exercise we started Thursday's (July 25) lecture with, just with a bit more guidance and a more "practical" final answer. Let $Y_1, \dots, Y_n \stackrel{\text{i.i.d.}}{\sim} \text{Exp}(\theta)$; we ultimately wish to construct a $(1 - \alpha) \times 100\%$ confidence interval for $\tau(\theta) := \theta^2$ (the population variance).

- Find $\hat{\tau}_{\text{MLE}}$, the maximum likelihood estimator for τ . You may use, without proof, the fact that the MLE for θ is given by \bar{Y}_n .
- Identify the asymptotic sampling distribution of $\hat{\tau}_{\text{MLE}}$, by appealing to the theorem titled "Theorem (Asymptotic MLE Result)" from lecture.
- Use the asymptotic normality of the maximum likelihood estimator to show that a theoretical large-sample $(1 - \alpha) \times 100\%$ confidence interval for τ is given by

$$\bar{Y}_n^2 \pm \Phi^{-1}\left(1 - \frac{\alpha}{2}\right) \cdot \frac{2\theta^2}{\sqrt{n}}$$

- Now, in practice, the interval posited in part (c) isn't really usable, because it depends on θ (whose value is unknown). As such, a more "practical" CI is obtained by replacing θ^2 with its MLE: $(\bar{Y}_n)^2$:

$$\bar{Y}_n^2 \pm \Phi^{-1}\left(1 - \frac{\alpha}{2}\right) \cdot \frac{2\bar{Y}_n^2}{\sqrt{n}}$$

(Though there is perhaps some question about whether or not replacing θ^2 with an estimator for θ^2 breaks the normality of our estimator, we won't concern ourselves with that for this course.)

Suppose we obtain an observed sample of 100 values from the $\text{Exp}(\theta)$ distribution, and it is calculated that $\bar{y}_{100} = 2.720$. Construct a 95% CI for θ^2 . Use a computer software (e.g. WolframAlpha, etc.) to compute any inverse-CDF values, so that you report only a numerical interval.

2. Consider $Y_1, \dots, Y_n \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$ where $\mu \in \mathbb{R}$ is unknown but $\sigma^2 > 0$ is known. In this problem, we will work toward constructing a lower-tailed test at an α level of significance.

- Let $H_0 : \mu = \mu_0$. Write down the correct alternative hypothesis for our test (remember that we're trying to construct a lower-tailed test).
- Should the rejection region be: "values of \bar{Y}_n that are much larger than μ_0 " or "values of \bar{Y}_n that are much smaller than μ_0 "? Justify your answer verbally.
- Show that the critical value for the test must be $c := \Phi^{-1}(\alpha)$, if our test statistic is $\frac{\bar{Y}_n - \mu_0}{\sigma/\sqrt{n}}$