LECTURE 2 (6/25) CHALKBOARD EXAMPLES

PSTAT 120B: Mathematical Statistics, I
Summer Session A, 2024 with Instructor: Ethan P. Marzban

Example 1

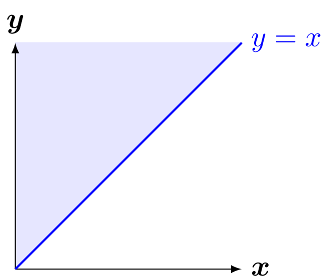
Suppose (X, Y) is a continuous bivariate random vector with joint p.d.f. given by

$$f_{X,Y}(x,y) = \begin{cases} \lambda^3 x e^{-\lambda y} & \text{if } 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $f_Y(y)$, the marginal density of Y.

(b) Find $f_{X|Y}(x \mid y)$, the conditional density of $(X \mid Y = y)$

Part (a)



$$f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x,y) \, \mathrm{d}x = \int_{-\infty}^{\infty} \lambda^3 x e^{-\lambda y} \cdot \mathbb{1}_{\{0<x<y\}} \cdot \mathbb{1}_{\{y \geq 0\}} \, \mathrm{d}x$$

$$= \lambda^3 e^{-\lambda y} \cdot \mathbb{1}_{\{y \geq 0\}} \cdot \int_0^y x \, \mathrm{d}x = \boxed{\frac{\lambda^3}{2} y^2 e^{-\lambda y} \cdot \mathbb{1}_{\{y \geq 0\}}}$$

Part (b)

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\lambda^3 x e^{-\lambda y} \cdot \mathbb{1}_{\{0<x<y\}} \cdot \mathbb{1}_{\{y \geq 0\}}}{\frac{\lambda^3}{2} y^2 e^{-\lambda y} \cdot \mathbb{1}_{\{y \geq 0\}}} = \boxed{\frac{2x}{y^2} \cdot \mathbb{1}_{\{0<x<y\}}}$$

To elaborate a bit on the numerator: essentially, I'm taking the indicator $\mathbb{1}_{\{0<x<y<\infty\}}$ and "breaking it up" by utilizing the fact that, for events E and F, $\mathbb{1}_E \cdot \mathbb{1}_F = \mathbb{1}_{E \cap F}$. So, I'm essentially saying: "if 0 is less than x which is less than y which is less than ∞", this is equivalent to saying "0 is less than x is less than y and y is positive". We don't need to include x in the second condition, since we've already included it in the first.

. . . .

Example 1 (cont'd)

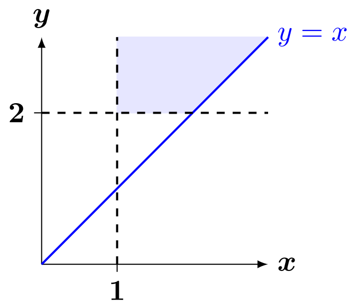
(c) Compute $\mathbb{P}(X \geq 1 \mid Y \geq 2)$

(d) Compute $\mathbb{P}(X \geq 1 \mid Y = 2)$

Part (c) Since we're conditioning on an event with nonzero probability, we can use the definition of conditional probability to write:

$$\mathbb{P}(X \geq 1 \mid Y \geq 2) = \frac{\mathbb{P}(X \geq 1 \, , \, Y \geq 2)}{\mathbb{P}(Y \geq 2)}$$

The numerator can be computed by double-integrating the joint density over the region



The order dx dy will be easier, to avoid splitting the integral into two.

$$\mathbb{P}(X \geq 1,\, Y \geq 2) = \int_2^\infty \int_1^y \lambda^3 x e^{-\lambda y}\, \mathrm{d}x\, \mathrm{d}y = \frac{1}{2}\int_2^\infty \lambda^3 (y^2 - 1) e^{-\lambda y}\, \mathrm{d}y$$
$$= \frac{1}{2}\left[\int_2^\infty \lambda^3 y^2\, \mathrm{d}y - \int_2^\infty \lambda^3 e^{-\lambda y}\right]$$

For the first integral, integrate by parts (tabularly):

$\lambda^3 y^2$	$+$	$e^{-\lambda y}$
$2\lambda^3 y$	$-$	$-\frac{1}{\lambda}e^{-\lambda y}$
$2\lambda^3$	$+$	$\frac{1}{\lambda^2}e^{-\lambda y}$
0		$-\frac{1}{\lambda^3}e^{-\lambda y}$

$$\implies \left(\lambda^2 y^2 + 2\lambda y + 2\right) e^{-\lambda y}\Big]_{y=\infty}^{y=2} = (4\lambda^2 + 4\lambda + 2)e^{-2\lambda}$$

The second integral is simply

$$\int_2^\infty \lambda^3 e^{-\lambda y}\, \mathrm{d}y = \lambda^2 \int_2^\infty \lambda e^{-\lambda y}\, \mathrm{d}y = \lambda^2 e^{-2\lambda}$$

Hence,

$$\mathbb{P}(X \geq 1,\, Y \geq 2) = \frac{1}{2}\left[\int_2^\infty \lambda^3 y^2\, \mathrm{d}y - \int_2^\infty \lambda^3 e^{-\lambda y}\right] = \frac{1}{2}\left[(4\lambda^2 + 4\lambda + 2)e^{-2\lambda} - \lambda^2 e^{-2\lambda}\right]$$
$$= \frac{3\lambda^2 + 4\lambda + 2}{2} \cdot e^{-2\lambda} = \left(\frac{3}{2}\lambda^2 + 2\lambda + 1\right) e^{-2\lambda}$$

For the denominator:

$$\mathbb{P}(Y \geq 2) = \int_2^\infty f_Y(y)\, \mathrm{d}y = \int_2^\infty \frac{\lambda^3}{2} y^2 e^{-\lambda y}\, \mathrm{d}y$$
$$= \frac{1}{2}(4\lambda^2 + 4\lambda + 2)e^{-2\lambda} = \left(2\lambda^2 + 2\lambda + 1\right) e^{-2\lambda}$$

and so, putting everything together,

$$\mathbb{P}(X \geq 1 \mid Y \geq 2) = \boxed{\frac{\frac{3}{2}\lambda^2 + 2\lambda + 1}{2\lambda^2 + 2\lambda + 1} = \frac{3\lambda^2 + 4\lambda + 2}{4\lambda^2 + 4\lambda + 2}}$$

Part (b) We cannot use the 120A definition of conditional probability, since the event we are conditioning on has zero probability. Instead, we must integrate the conditional density.

$$\begin{aligned} f_{X|Y}(x \mid 2) &= \frac{2x}{2^2} \cdot \mathbb{1}_{\{0<x<2\}} = \frac{x}{2} \cdot \mathbb{1}_{\{0<x<2\}} \\ \mathbb{P}(X \geq 1 \mid Y = 2) &= \int_1^\infty f_{X|Y}(x \mid y)\, \mathsf{d}x \\ &= \int_1^\infty \frac{x}{2} \cdot \mathbb{1}_{\{0<x<2\}}\, \mathsf{d}x = \frac{1}{2}\int_1^2 x\, \mathsf{d}x = \frac{1}{4}(4-1) = \boxed{\frac{3}{4}} \end{aligned}$$