LECTURE 3 (6/26) CHALKBOARD EXAMPLES

PSTAT 120B: Mathematical Statistics, I
Summer Session A, 2024 with Instructor: Ethan P. Marzban

Example 1

Let $(X|Y = y) \sim$ Bin$(y, 0.25)$ and $Y \sim$ Pois(2). What is $\mathbb{E}[X]$?

Since the expectation of the Bin(n, p) distribution is just np, the problem statement allows us to directly conclude

$$\mathbb{E}[X \mid Y = y] = y \cdot 0.25 = 0.25y$$

Hence, by our two-step procedure for computing $\mathbb{E}[X \mid Y]$ (which tells us to first compute $\mathbb{E}[X \mid Y = y]$ and then replace y with Y), we have

$$\mathbb{E}[X \mid Y] = 0.25Y$$

Finally, by the Law of Iterated Expectations,

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]] = \mathbb{E}[0.25Y] = 0.25 \cdot \mathbb{E}[Y] = 0.25 \cdot 2 = 0.5$$

Example 2

Suppose I roll a fair six-sided die. Then, whatever number the die lands on, I flip that many fair coins. Let X denote the number of heads. Compute $\mathbb{E}[X]$

Back in Lecture 1 we derived a formula for the PMF of X, so we could try and appeal to the definition of expectation: $\mathbb{E}[X] := \sum_x p_X(x)$. However, the resulting sum will be very difficult to evaluate directly. Instead, let's appeal to the Law of Iterated Expectations. Specifically, let N denote the result of the die roll: then

$$N \sim \text{DiscUnif}\{1, 2, \cdots, 6\}$$

Furthermore, as was argued before,

$$(X \mid N = n) \sim \text{Bin}(n, 1/2)$$

i.e. *given* that we tossed n coins, X (which counts the number of heads) follows a binomial distribution. Hence,

$$\mathbb{E}[X \mid N = n] = n \cdot (1/2) = \frac{n}{2} \implies \mathbb{E}[X \mid N] = \frac{N}{2}$$

and so, by the Law of Iterated Expectations,

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid N]] = \mathbb{E}\left[\frac{N}{2}\right] = \frac{1}{2}\mathbb{E}[N] = \frac{1}{2} \cdot \frac{7}{2} = \frac{7}{4}$$