Supplemental: Monotonicity

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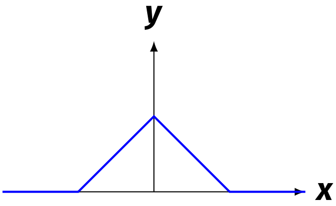
Functions

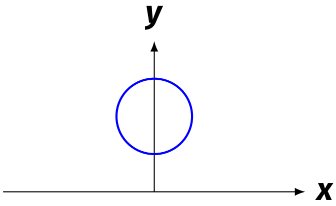
- As you can imagine - since “transformations” is just a fancy way of saying “functions of random variables,” it will be incredibly useful to understand some key properties of functions.
- Now, the review we will embark on over the next few slides is neither complete nor comprehensive! But, it should (hopefully) serve to jog your memories on some material you (hopefully) saw back in precalculus.

Functions

- One of the first things to remember about functions is that they, definitionally, cannot map a single input to multiple outputs.
- Graphically, this leads to something known as the **vertical line test**: a given graph is the graph of a function if and only if any vertical line drawn on the cartesian plane intersects the graph at no more than 1 point.



A Valid Function



Not a Valid Function

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Inverse Functions

- The **inverse** of a function $g : \mathbb{R} \to \mathbb{R}$, if it exists, is the (unique) function $g^{-1}(\cdot)$ such that

$$(\forall x \in \mathbb{R})[g^{-1}(g(x)) = g(g^{-1}(x)) = x]$$

- By the way, you'll see me use the symbol $\forall$ (and the related symbol $\exists$) from time to time in this class.
- $\forall$ is known as the **universal quantifier**, and is the proper mathematical notation for "for all" or "for every".
- $\exists$ is known as the **existential quantifier**, and is the proper mathematical notation for "there exists" or "for at least one".

Inverse Functions

- Algorithmically, we find the inverse of a function $g(x)$ by writing $x = g(y)$, and solving for y.
- For example, let's say we want to find the inverse of the function $g(x) = 3x + 1$.
  - To do so, we write $x = 3y + 1$, which then allows us to solve for y in terms of x:

$$y = \frac{x - 1}{3} =: g^{-1}(x)$$

- Pictorially, inverting a function amounts to reflecting its graph about the line $y = x$.

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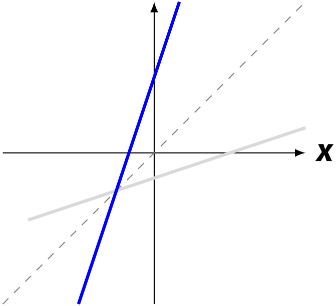
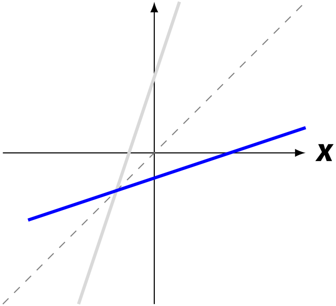
Inverse Functions

$y = g(x)$



$g(x) = 3x + 1$

$y = g^{-1}(x)$



$g^{-1}(x) = \frac{x-1}{3}$

Inverse Functions

- Now, this pictorial view of inverting a function (i.e. reflecting about the line $y = x$) allows us to see when a given function will not have an inverse.
  - Such a function is often referred to as **noninvertible**.

- Specifically, you may remember that a function is invertible if and only if its graph passes the **horizontal line test**: any *horizontal* line drawn on the cartesian plane must intersect the graph at no more than 1 point.

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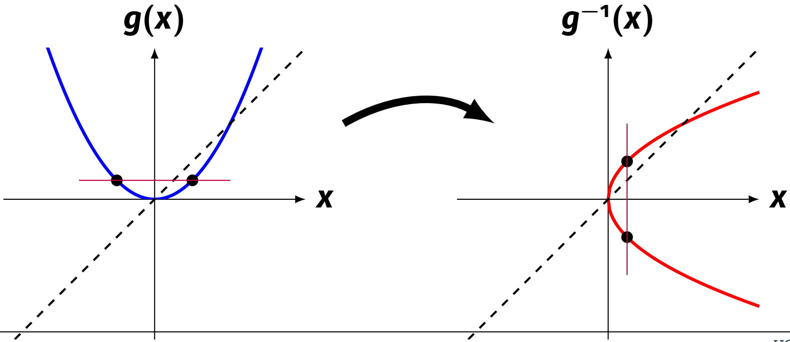
Inverse Functions

- Suppose the graph of a function $g(\cdot)$ fails the horizontal line test at a given point x'. Then, when we reflect the graph of $g(x)$ about the line x, the resulting graph will fail the *vertical* line test at x' and therefore not even be a valid function!



Monotonicity

- Finally, there are a few terms that we will use quite often throughout the next few lectures, so allow me to pose them as proper definitions.

Definition (Monotonicity)

A function $g : \mathbb{R} \to \mathbb{R}$ is said to be:

- **monotonically increasing** if, for any $x \leq y$ we have $g(x) \leq g(y)$
- **monotonically decreasing** if, for any $x \leq y$ we have $g(x) \geq g(y)$.

Any function that is either monotonically increasing or monotonically decreasing is sometimes said to simply be **monotonic**.

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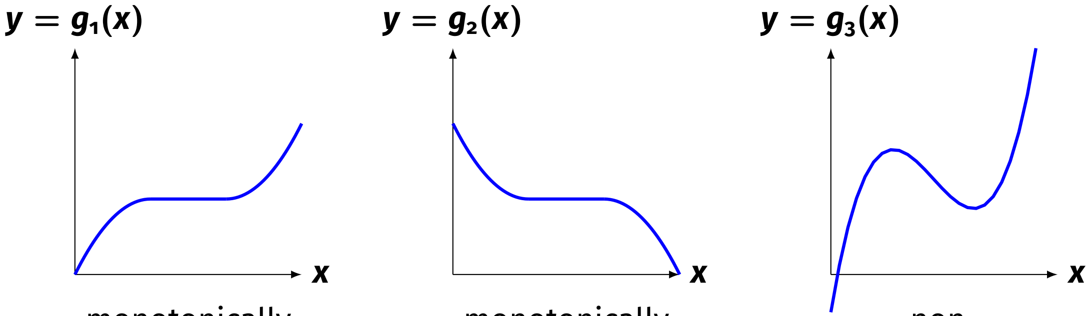
Monotonicity



monotonically increasing

monotonically decreasing

non-monotonic

Strict Monotonicity

Definition (Strict Monotonicity)

A function $g : \mathbb{R} \to \mathbb{R}$ is said to be:

- **strictly increasing** if, for any $x < y$ we have $g(x) < g(y)$
- **monotonically decreasing** if, for any $x < y$ we have $g(x) > g(y)$.

Any function that is either strictly increasing or strictly decreasing is sometimes said to simply be **strictly monotonic**.

- So, as an example, neither $g_1(\cdot)$, $g_2(\cdot)$, nor $g_3(\cdot)$ [on the previous slide] are strictly monotonic.

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Strict vs. Non-Strict Monotonicity

- The main difference between strict and non-strict monotonicity is whether we allow the function to be constant over a portion of the real line or not.
- Monotonic functions *may* be constant over portions of the real line, whereas strictly monotonic functions *cannot*- they are *always* either increasing or decreasing, never staying flat.
- Another way of saying this is that strict monotonicity is a *stronger* condition than monotonicity - all strictly monotonic functions are monotonic, but not all monotonic functions are strictly monotonic.

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Monotonicity and Invertibility

- So why did I introduce these terms? Well, there are a couple of reasons.
- One reason is that we can perhaps see how monotonicity ties into *invertibility* - strictly monotonic functions are *always* invertible!
    - The same cannot be said about monotonic functions that are not strictly monotonic - remember that a monotonic (but not strictly monotonic) function is constant over at least one portion of the real line, and hence its graph will fail the horizontal line test over this region.
    - If it helps, I encourage you to take a look at the graphs of the functions $g_1(\cdot)$ and $g_2(\cdot)$ from a few slides ago.

Monotonicity and Inequalities

- Monotonicity has some interesting tie-ins with inequalities as well!

Theorem Transforming Inequalities

Consider two points $x, y \in \mathbb{R}$ with $x \leq y$, and a function $g : \mathbb{R} \to \mathbb{R}$.

(I) If $g(\cdot)$ is monotonically increasing, then $g(x) \leq g(y)$.

(II) If $g(\cdot)$ is monotonically decreasing, then $g(x) \geq g(y)$.

Monotonicity

- Finally, I should note that functions that are not monotonic over the entire real line may still be monotonic over a restricted domain.
- For example, the function $g(x) = x^2$ is *not* monotonic over $\mathbb{R}$ (try sketching a graph to convince yourself of this).
- However, if we restrict $g(x) = x^2$ to the positive real line (i.e. we effectively ignore the left-half of the parabola), the resulting function *is* strictly monotonic.
- This is a fact we will leverage a fair amount in this class.