Topic 3: Estimation

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Outline

1. Unbiasedness, and MSE
2. Other Assessments

Unbiasedness, and MSE

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Recap

Goal

Given a population, from which random variables are assumed to follow a distribution $\mathcal{F}$ with parameter θ, we seek to take random samples $\vec{\boldsymbol{Y}} := (Y_1, \cdots, Y_n)$ from this population and use them to estimate the true value of θ.

- **Estimator** $\widehat{\theta}_n$: a statistic being used to estimate θ.
  - Alternatively, "a rule, often expressed as a formula, that tells how to calculate the value of an estimate based on the measurements contained in a sample."
- **Estimate**: an observed instance of our estimator.

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Recap

- For instance, last lecture we talked about trying to estimate a population mean μ.
- Given a sample $Y_1, \cdots, Y_n$ from the population (which, again, has mean μ), we can consider several different estimators for μ:
  - $\widehat{\mu}_{n,1} := \overline{Y}_n := n^{-1} \sum_{i=1}^{n} Y_i$
  - $\widehat{\mu}_{n,2} := (Y_1 + Y_3)/2$
  - $\widehat{\mu}_{n,3} := Y_5$
- Since there are many potential estimators we can use to estimate a parameter, we'd like to determine how to quantify how "well" an estimator is performing.

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Recap:

- One metric we talked about was that of **bias**, which is the signed distance between the expected value of our estimator and the true parameter value:

$$\text{Bias}(\widehat{\theta}_n\ ,\ \theta) := \mathbb{E}[\widehat{\theta}_n] - \theta$$

- An **unbiased** estimator $\widehat{\theta}_n$ of θ is one that satisfies $\mathbb{E}[\widehat{\theta}_n] = \theta$.
  - I.e., an unbiased estimator "gets it right on average."

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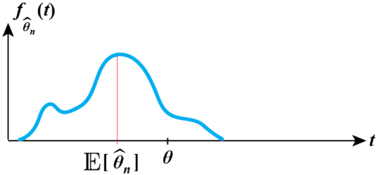
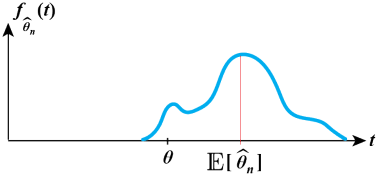
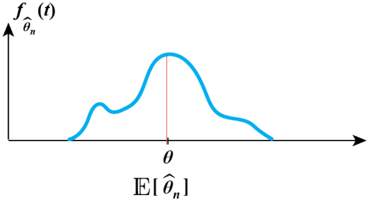
Bias



NEGATIVELY BIASED



POSITIVELY BIASED



UNBIASED

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Recap

- I also introduced an analogy our textbook uses, whereby we can think of estimation as trying to hit a target with a revolver.
- The bullseye/target is the parameter we're trying to estimate; every bullet we fire is an estimate, and our shooting prowess is essentially the estimator.
- Assessing how well an estimator is performing is, then, akin to assessing how good of a shot we are!

Recap

- An unbiased estimator is akin to a marksman who, on average, hits the target.
- More specifically, an unbiased estimator is akin to a marksperson whose average location of many shots is right on the target.

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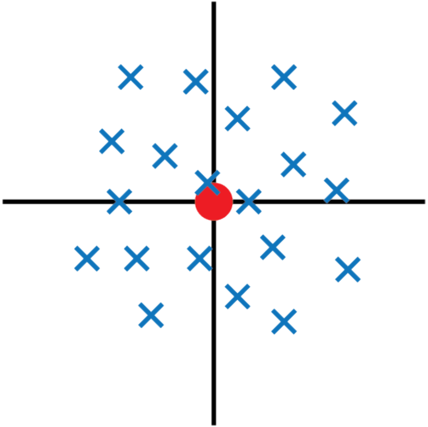
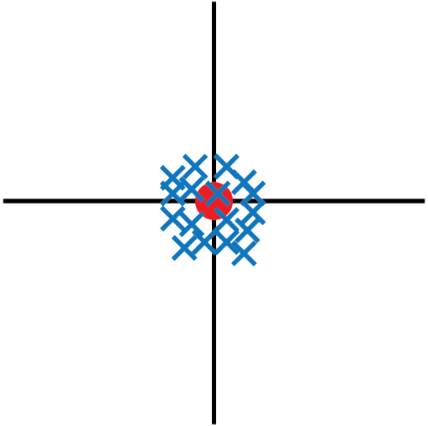
Unbiasedness



- This marksperson is an example of an unbiased estimator - the average location of all of their shots (depicted as blue $\times$'s) is quite close to the target (indicated in red).
- But would we classify them as a "*good*" marksperson? Specifically, how would we classify their performance in comparison to...

Unbiasedness



- This marksperson is an *also* “unbiased”.
- But doesn’t our intuition tell us that they are performing “better” than the marksperson on the previous slide?

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Precision vs. Accuracy

- So, this perhaps indicates to us that unbiasedness alone, though a decent critera to strive for, isn't the whole picture.
- Indeed, this relates to the distinction between two very important concepts in science (not just statistics): **precision** vs **accuracy**.
- Accuracy, more or less, corresponds to our notion of unbiasedness - it refers to "on average, how close are we to the ground truth?"
- *Precision* is the other half of the story that we're missing - it relates to "on average, how much *variability* is there from trial to trial?"

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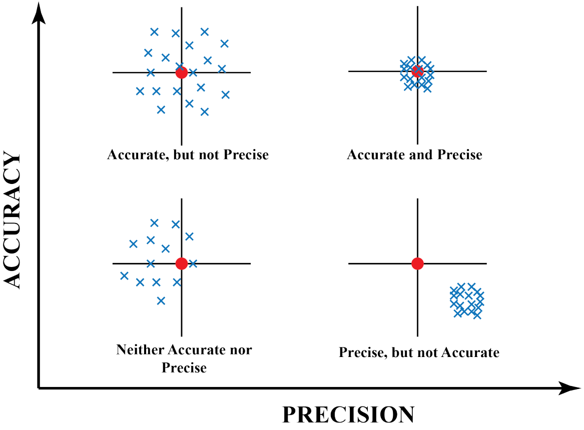
Precision vs. Accuracy



Precision

- As was hinted at before, *precision* is linked (in the context of estimation) to the *variance* of a given estimator.
- Not only would we like our estimator to get the right value of θ on average, we'd also like to be fairly certain that on any particular draw we're close to the true value!

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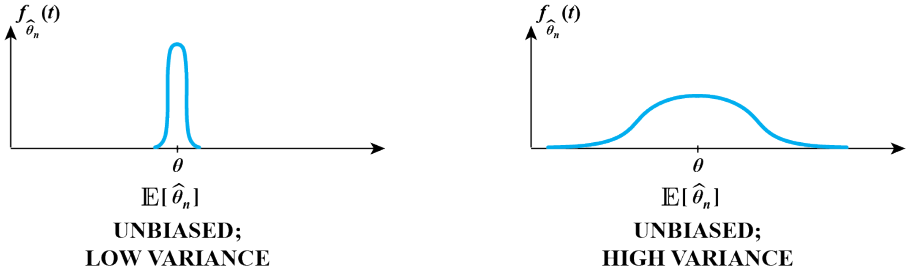
Precision


$f_{\widehat{\theta}_n}(t)$
θ
$\mathbb{E}[\widehat{\theta}_n]$
UNBIASED;
LOW VARIANCE
$f_{\widehat{\theta}_n}(t)$
θ
$\mathbb{E}[\widehat{\theta}_n]$
UNBIASED;
HIGH VARIANCE

Ideal Estimator

- So, based on everything we've discussed so far, it seems as though an "ideal" estimator is one that is both unbiased and also possesses a small variance.
- Thankfully, we have a metric that is able to simultaneously assess a given estimator's bias and variance - this metric is called the **mean square error** (MSE).

Definition (MSE)

The mean square error (MSE) of an estimator $\widehat{\theta}_n$ for a parameter θ is defined to be

$$\mathrm{MSE}(\widehat{\theta}_n\ ,\ \theta) := \mathbb{E}\left[\left(\widehat{\theta}_n - \theta\right)^2\right]$$

Bias-Variance Decomposition

Theorem (Bias-Variance Decomposition)

Given an estimator $\widehat{\theta}_n$ for a parameter θ, we have that

$$\mathrm{MSE}(\widehat{\theta}_n, \theta) = \left[\mathrm{Bias}(\widehat{\theta}_n, \theta)\right]^2 + \mathrm{Var}(\widehat{\theta}_n)$$

- We'll save the proof for later.

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Bias-Variance Decomposition

Theorem (MSE of an Unbiased Estimator)

Given an unbiased estimator $\widehat{\theta}_n$ for a parameter θ, we have that

$$\mathrm{MSE}(\widehat{\theta}_n, \theta) = \mathrm{Var}(\widehat{\theta}_n)$$

- This follows directly from the Bias-Variance Decomposition, along with the definition of unbiasedness.

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Example

Example

Let $Y_1, \cdots, Y_n \stackrel{\text{i.i.d.}}{\sim} \text{Unif}[0, \theta]$ for some deterministic constant $\theta > 0$. Compute the mean square error of using $\overline{Y}_n$ as an estimator for θ.

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Solutions

- When trying to compute the MSE of a given estimator, it's usually a good idea to start off by computing the expected value of the estimator.
- We know that the expected value of the sample mean is the population mean, which in this case is $(\theta + 0)/2 = \theta/2$ [we get this from the formula for the expectation of the Uniform distribution]. Hence,
$$\mathbb{E}[\overline{Y}_n] = \frac{\theta}{2}$$

Solutions

- Let's now compute the bias of using $\overline{Y}_n$ as an estimator for θ. By definition,
$$\text{Bias}(\overline{Y}_n, \theta) = \mathbb{E}[\overline{Y}_n] - \theta = \frac{\theta}{2} - \theta = -\frac{\theta}{2}$$
- Finally, we can compute the variance of $\overline{Y}_n$:

$$\text{Var}(\overline{Y}_n) = \frac{\text{Var}(Y_1)}{n} = \frac{\left(\frac{\theta^2}{12}\right)}{n} = \frac{\theta^2}{12n}$$

Solutions

- So, by the Bias-Variance Decomposition,

$$\begin{aligned}
\mathrm{MSE}(\overline{Y}_n, \theta) &= \left[\mathrm{Bias}(\widehat{\theta}_n, \theta)\right]^2 + \mathrm{Var}(\widehat{\theta}_n) \\
&= \left(-\frac{\theta}{2}\right)^2 + \frac{\theta^2}{12n} = \boxed{\frac{\theta^2(3n+1)}{12n}}
\end{aligned}$$

Clicker Question

Clicker Question 1

Which of the following statements is true?

(A) An ideal estimator has a very large MSE

(B) An ideal estimator has an MSE that is very close to 0

(C) An ideal estimator has an MSE that is very negative

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Clicker Question

Clicker Question 2

Consider $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, 1)$, and further consider the following two estimators of μ:

$$\widehat{\mu}_{n,1} := \frac{Y_1 + Y_2}{2}; \qquad \widehat{\mu}_{n,2} = \overline{Y}_n$$

In terms of MSE, which (if either) estimator performs better?

(A) $\widehat{\mu}_{n,1}$

(B) $\widehat{\mu}_{n,2}$

(C) The two estimators perform equally well in terms of MSE

Result

Theorem (Sample Variance is an U.B.E. for Population Variance)

Given an i.i.d. sample $\{Y_i\}_{i=1}^n$ from a distribution with unknown variance σ^2, then

$$S_n^2 := \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \overline{Y}_n)^2$$

is an unbiased estimator for σ^2.

Example

Example

Given $Y_1, \cdots, Y_n \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma^2)$ for some unknown $\sigma^2 > 0$, compute the MSE of using

$$S_n^2 := \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \overline{Y}_n)^2$$

as an estimator for σ^2.

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Solutions

- Since S_n^2 is an unbiased estimator for σ^2 (by the previous theorem), we know (by the theorem pertaining to the MSE of an unbiased estimator) that $\mathrm{MSE}(S_n^2, \sigma^2) = \mathrm{Var}(S_n^2)$.
- By the result pertaining to the sampling distribution of S_n^2 (mentioned a few lectures ago), we have

$$\frac{n-1}{\sigma^2} S_n^2 \sim \chi^2_{n-1} \sim \mathrm{Gamma}\left(\frac{n-1}{2}, \ 2 \right)$$

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Interstitial Result

Theorem (Closure of Gamma Distribution under Scalar Multiplication)

Given $Y \sim \text{Gamma}(\alpha, \beta)$ and $U := (cY)$ for some $c > 0$, we have that $U \sim \text{Gamma}(\alpha, c\beta)$.

Proof.
Use the MGF method. $\square$

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Solutions

$$\frac{n-1}{\sigma^2} S_n^2 \sim \text{Gamma}\left(\frac{n-1}{2}, 2 \right)$$

$$\implies S_n^2 \sim \text{Gamma}\left(\frac{n-1}{2}, 2 \cdot \frac{\sigma^2}{n-1} \right)$$

$$\implies \text{Var}(S_n^2) = \left(\frac{n-1}{2} \right) \cdot \left(2 \cdot \frac{\sigma^2}{n-1} \right)^2$$

$$= \frac{n-1}{2} \cdot \frac{4\sigma^4}{(n-1)^2} = \boxed{\frac{2\sigma^4}{n-1}}$$

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Other Assessments

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Leadup

- MSE is a very useful metric for measuring how well a given estimator is performing!
- Indeed, as we've seen, it even allows us to compare the performance of two estimators, by simply comparing their MSE's (remember the result of our clicker questions!)
- But, there are other properties we might seek to impose on our estimators.

Leadup

- Recall last lecture that I introduced the notion of an *asymptotically* unbiased estimator.
  - As a review, an estimator $\widehat{\theta}_n$ for a parameter θ is said to be asymptotically unbiased if
$$\lim_{n \to \infty} \mathrm{Bias}(\widehat{\theta}_n, \theta) = 0$$
- Indeed, the field of **asymptotics** is the subfield of statistics dedicated to studying what happens as our sample size (n) becomes very large.
- Borrowing from asymptotics, we may seek to impose certain *large-sample* properties we would like our "good" estimators to obey.

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Disclaimer

- Disclaimer - things are about to get pretty math-y.
- I'll do my best to translate these results into words - I urge you to think through these definitions carefully on your own!

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Consistency

Definition (Consistent Estimator)

An estimator $\widehat{\theta}_n$ is said to be a <u>**consistent**</u> estimator for θ if

$$\widehat{\theta}_n \xrightarrow{p} \theta$$

That is, if either of the two equivalent conditions hold for any $\varepsilon > 0$:

$$\lim_{n \to \infty} \mathbb{P}(|\widehat{\theta}_n - \theta| \geq \varepsilon) = 0$$

$$\lim_{n \to \infty} \mathbb{P}(|\widehat{\theta}_n - \theta| < \varepsilon) = 1$$

Interpretation

- Okay, what the heck is this saying???
- Let's parse through the first definition:

$$\lim_{n \to \infty} \mathbb{P}(|\widehat{\theta}_n - \theta| \geq \varepsilon) = 0$$

- What is the event $\{|\widehat{\theta}_n - \theta| \geq \varepsilon\}$ saying?
- Well, $|\widehat{\theta}_n - \theta|$ is essentially the distance between $\widehat{\theta}_n$ and θ.
- Hence, the event $\{|\widehat{\theta}_n - \theta| \geq \varepsilon\}$ is essentially just the event "$\widehat{\theta}_n$ is very far away from θ."

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Interpretation

- Therefore, $\mathbb{P}(|\widehat{\theta}_n - \theta| \geq \varepsilon)$ is just the probability that $\widehat{\theta}_n$ is very far away from θ.
- What the definition of consistency is saying is: this probability goes to zero as our sample size increases.
- Equivalently, $\mathbb{P}(|\widehat{\theta}_n - \theta| \geq \varepsilon)$ is just the probability that $\widehat{\theta}_n$ is very close to θ.
- The definition of consistency also asserts that this probability goes to 1 as our sample size increases.

Interpretation

- So, all in all, consistency is saying: as we keep taking samples of larger and larger size, we become more and more *certain* that $\widehat{\theta}_n$ is very close to θ.
- That sounds like a pretty desirable property for an estimator to have, doesn't it?

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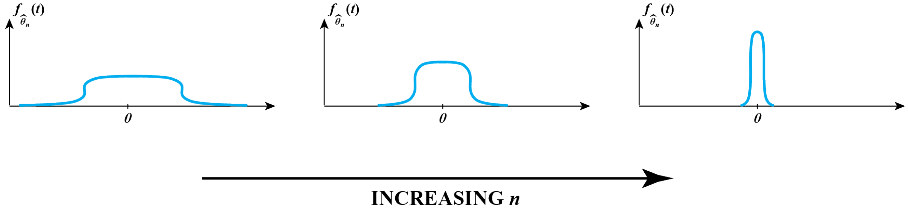
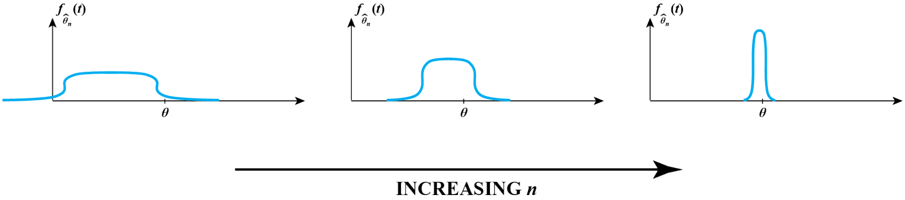
Consistent and Unbiased



- This is a (cartoon) example of an estimator that is unbiased *and* consistent.
- There do exist consistent estimators that are biased:

Consistent yet Biased



- You can (and will) show that S_n, the sample standard deviation, is a biased yet consistent estimator for σ, the population standard deviation.
- Fun fact - the background of our course logo contains an example of a biased yet consistent estimator!

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