SOME PSTAT 120A-STYLE REVIEW PROBLEMS

PSTAT 120B: Mathematical Statistics, I
Summer Session A, 2024 with Instructor: Ethan P. Marzban

1. Wait times (in minutes) at *Cajé* are uniformly distributed between 5 minutes and 15 minutes. Suppose a customer is selected at random from *Cajé*, and their wait time is recorded.
   (a) What is the probability that the selected customer waits fewer than 7 minutes?
   (b) What is the expected amount of time this customer should expect to wait in line?
   (c) Given that the customer has already waited for 7 minutes but hasn't been served yet, what is the probability that they end up waiting for more than 10 minutes?

2. Luna the Golden Retriever has buried a bone somewhere in the backyard. Unfortunately, she can't quite remember where she buried it! As such, she keepds digging holes in the hopes of finding her bone - once she finds her bone, she stops digging holes. Suppose each hole Luna digs has a 25% chance of containing her bone, independently of all other holes.
   (a) If X denotes the total number of holes Luna digs (including the final successful hole) before stopping, what distribution does X follow? Include both a distribution name as well as any/all relevant parameter(s).
   (b) What is the probability that Luna has to dig 20 or more holes before finding her bone?
   (c) What is the probability that Luna has to dig 19 or fewer holes before finding her bone?

3. Let X be a random variable, and let $a, b \in \mathbb{R}$ be deterministic constants. Use first principles to prove that expectations are **linear**; that is, $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$. As a hint: consider the discrete and continuous cases separately, and start each case by applying the LOTUS and leveraging linearity of sums and expectations. (You may assume that X is <u>not</u> mixed.)

4. The Celestial Toymaker[1] has decided to play a game with me. On a table, he lines up an infinite number of boxes (he *is* the god of games, after all). With probability $(1/2)^i$ he selects box number i [where $i = 1, 2, 3, \cdots$]. Inside box number i there are 3^i marbles, one of which is red and the remainder of which are blue. So, for example, box 1 is selected with probability (1/2), and contains 1 red marble and 2 blue marbles; box 2 is selected with probability (1/4), and contains 1 red marble and 8 blue marbles, etc. The Toymaker selects a box, and then draws a marble.
   (a) What is the probability that the Toymaker selects a red marble?
   (b) Given that the Toymaker selected a red marble, what is the problem that he drew from box 4?

5. Consider a sequence $\{X_i\}_{i=1}^n$ of i.i.d. random variables with common mean μ and common variance σ^2. Define
$$\overline{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i$$
to be the sample mean. Compute Corr$(X_1, \overline{X}_n)$, the correlation between X_1 and the sample mean. **Hint:** Bilinearity.

[1] If you're curious, this is a character from the British Sci-Fi teleision show *Doctor Who*.

6. Let $(X, Y) \sim f_{X,Y}$. Prove that

$$\mathbb{E}[X] = \iint_{\mathbb{R}^2} x f_{X,Y}(x, y) \, \mathrm{d}A$$

Hint: Iterate the integral on the RHS, leverage the relationship between marginal densities and joint densities, and finally recognize the definition of $\mathbb{E}[X]$.

7. Let $(X, Y) \sim f_{X,Y}$ where

$$f_{X,Y}(x, y) = k(1 - y) \cdot \mathbb{1}_{\{0 \leq x \leq y \leq 1\}}$$

   (a) Find the value of k that ensures this is a valid joint density function.

   (b) Compute $\mathbb{P}(X \leq 3/4 \, , \, Y \geq 1/2)$.

8. Let $X \sim$ Unif$[a, b]$.

   (a) Show that X has MGF given by

$$M_X(t) = \begin{cases} \frac{e^{tb} - e^{ta}}{t(b-a)} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$$

   Be careful with the cases you consider!

   (b) Is the above MGF continuous at $t = 0$? (Recall that this question is important as a lot of our MGF-related results assume continuity in an interval containing $t = 0$!)

   (c) Derive a simple closed-form expression for

$$\left. \frac{\mathrm{d}}{\mathrm{d}t^n} \right|_{t=0} \left[\frac{e^{tb} - e^{ta}}{t} \right]$$

   where the notation $\frac{\mathrm{d}}{\mathrm{d}t^n}\big|_{t=0}$ means "the n^{th} derivative, evaluated at $t = 0$."