MIDTERM 1 REVIEW PROBLEMS

PSTAT 120B: Mathematical Statistics, I
Summer Session A, 2024 with Instructor: Ethan P. Marzban

1. Let (X, Y) be a continuous bivariate random vector with joint p.d.f. given by

$$f_{X,Y}(x, y) = \begin{cases} c \cdot xy & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

   where $c > 0$ is an as-of-yet undetermined constant.

   (a) Find the value of c that ensures this is a valid density function.

   (b) Find $f_Y(y)$, the marginal p.d.f. of Y.

   (c) Find $f_{X|Y}(x \mid y)$, the conditional density of X given $Y = y$.

   (d) Compute $\mathbb{E}[X]$ using the Law of Iterated Expectations.

   (e) Find $f_X(x)$, and verify your answer to part (d).

2. In each of the following parts, you will be provided with the conditional distribution of $(X \mid Y)$ and the marginal distribution Y. Using the provided information, compute $\mathbb{E}[X]$.

   (a) $(X \mid Y = y) \sim \text{Bin}(y, p); \quad Y \sim \text{Pois}(\mu)$

   (b) $(X \mid Y = y) \sim \text{Exp}(y); \quad Y \sim \text{Gamma}(\alpha, \beta)$

3. Let $(Y_1 \mid Y_2 = y_2) \sim \text{Exp}(1/y_2)$ and $Y_2 \sim \text{Exp}(\beta)$. Find an expression for $f_{Y_1}(y_1)$, the marginal density of Y_1. Be sure to include the support of Y_1!

4. Let Y_1 and Y_2 have the joint probability density function given by

$$f_{Y_1,Y_2}(y_1, y_2) = 6(1 - y_2) \cdot \mathbb{1}_{\{0 \le y_1 \le y_2 \le 1\}}$$

   (a) Find the marginal density functions for Y_1 and Y_2.

   (b) Compute $\mathbb{P}(Y_2 \le 1/2 \mid Y_1 \le 3/4)$.

   (c) Find $f_{Y_1|Y_2}(y_1 \mid y_2)$, and clearly specify the values of y_2 for which it is defined.

   (d) Find $f_{Y_2|Y_1}(y_2 \mid y_1)$, and clearly specify the values of y_1 for which it is defined.

   (e) Compute $\mathbb{P}(Y_2 \ge 3/4 \mid Y_1 = 1/2)$.

   (f) Compute $\mathbb{E}[Y_1 \mid Y_2]$.

5. Let $X, Y \overset{\text{i.i.d.}}{\sim} \text{Pois}(\lambda)$.

   (a) Show that $(X + Y) \sim \text{Pois}(2\lambda)$.

   (b) Identify the distribution of $(X \mid X + Y = n)$ for $n \in \mathbb{N}$ by name, including any/all relevant parameters

6. The waiting time Y until delivery of a new component for an industrial operation is uniformly distributed over the interval from 1 to 5 days. The cost of this delay is given by $U = 2Y^2 + 3$. Find the probability density function for U using any of the methods we discussed in lecture.

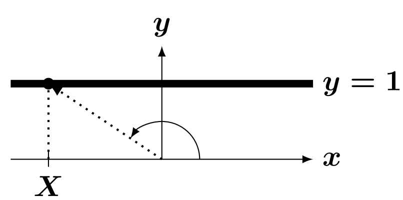
7. Let $Y \sim$ Unif$[0, 1]$ and define $U := aY + b$ for a constants $a > 0$ and $b \in \mathbb{R}$. Does U follow the uniform distribution? Justify your answer.

8. Suppose that Y has a gamma distribution with $\alpha = n/2$ for some positive integer $n \in \mathbb{N}$ and β equal to some specified value. Use the method of moment-generating functions to show that $W = 2Y/\beta$ has a χ_n^2 distribution.

9. A parachutist wants to land at a target T, but she finds that she is equally likely to land at any point on a straight line (A, B), of which T is the midpoint. Find the probability density function of the distance between her landing point and the target. [**Hint:** Denote A by –1, B by $+1$, and T by 0. Then the parachutist's landing point has a coordinate X, which is uniformly distributed between -1 and $+1$. The distance between X and T is $|X|$.]

10. Let $Y \sim \mathcal{N}(0, 1)$ and define the random variable U as $U := e^{\sigma Y + \mu}$ for constants $\sigma > 0$ and $\mu \in \mathbb{R}$.
    (a) Derive an expression for the density of U. **As An Aside:** the distribution of U is called the **lognormal distribution**.
    (b) We define the **median** of a continuous distribution with density $f_X(x)$ to be the value m such that $\mathbb{P}(X \leq m) = \mathbb{P}(X > m) = 1/2$. Show that the median of the lognormal distribution is e^{μ}. (You may use, without proof, the fact that the median of the standard normal distribution is 0.)

11. A particle is fired from the origin in a random (i.e. uniformly-distributed) direction pointing somewhere in the first two quadrants. The particle travels in a straight line, unobstructed, until it collides with an infinite wall located at $y = 1$. Let X denote the $x-$coordinate of the point of collision.



    (a) What is the expected value of the $x-$coordinate of the point of collision? **Do NOT first find the p.d.f. of X.**
    (b) Find $f_X(x)$, the probability density function (p.d.f.) of X
    (c) Confirm your answer to part (a) using your answer to part (b).

12. (**Challenge**) Let $X \sim$ Exp$(1/\lambda)$, and define $Y := \lceil X \rceil$. Identify the distribution of Y **by name**, taking care to include any/all relevant parameter(s). Recall that

$$\lceil x \rceil := \text{smallest integer larger than or equal to } x$$

so, for instance, $\lceil \pi \rceil = 4$. **Hint:** Identify appropriate values for a and b such that

$$\{\lceil X \rceil = y\} = \{a < X \leq b\}$$

Also, when you are trying to identify the name of the resulting distribution, I recommend pattern-matching with the support and form of the PMF. Once you have done so, you can do a quick computation to compute $\mathbb{E}[Y]$ to help you "cheat" and factorize the PMF into a more recognizable form.

13. (**Challenge**) Two points P_1 and P_2 are picked uniformly at random from the portion of the unit circle lying in the first quadrant. Let L denote the length of the chord connecting these two points: find $F_L(\ell)$, the density of L.

    Hint: note that, since the two points are selected uniformly at random in the first quadrant, the angle Θ between radii subtended by these points is uniformly distributed between $[0, \pi/2)$. Hence, sketch a picture and use the Law of Cosines to derive a formula relating L to U; then you can use one of the methods discussed in lecture to derive the desired density.