MIDTERM 2 REVIEW PROBLEMS

PSTAT 120B: Mathematical Statistics, I
Summer Session A, 2024 with Instructor: Ethan P. Marzban

1. Let Y_1, Y_2 be i.i.d. random variables with the following density:

$$f_Y(y) = 2y \cdot \mathbb{1}_{\{0 \leq y \leq 1\}}$$

   (a) Let $U_1 := Y_1 Y_2$. Find $f_{U_1}(u)$, the density of U_1.
   (b) Let $U_2 := Y_1 / Y_2$. Find $f_{U_2}(u)$, the density of U_1.
   (c) Let $U_3 := Y_1 + Y_2$. Find $f_{U_3}(u)$, the density of U_1.
   (d) Find $f_{Y_{(1)}}(y)$, the density of the first order statistic.
   (e) Find $f_{Y_{(n)}}(y)$, the density of the n^{th}order statistic.

2. ~~Let $Y_1, Y_2 \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$, and define $U := Y_2 / Y_1$.~~
   (a) ~~Express $F_U(u)$, the CDF of U at a point u, as a double integral. Do **not** evaluate this integral directly~~
   (b) ~~Use your answer to part (a) to find $f_U(u)$, the density of U. You will need to use the Fundamental Theorem of Calculus, Part I.~~
   (c) ~~Does $\mathbb{E}[U]$ exist? Why or why not?~~
   (d) ~~**(Challenge)** Recognize the distribution of U by name, taking care to include any/all relevant parameter. **Hint:** Focus on the variable part.~~

3. Let $Y_1, \cdots, Y_9 \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$.
   (a) Let $U_1 := Y_1 + \cdots + Y_9$. Identify the distribution of U_1 by name, taking care to include any/all relevant parameter(s).
   (b) Let $U_2 := Y_1^2 + \cdots + Y_9^2$. Identify the distribution of U_2 by name, taking care to include any/all relevant parameter(s).
   (c) Let $U_3 := \sum_{i=1}^{9} (Y_i - \overline{Y}_9)^2$. Identify the distribution of U_3 by name, taking care to include any/all relevant parameter(s).
   (d) Let

$$U_4 := \frac{U_1 \sqrt{8}}{3\sqrt{U_3}}$$

   where U_1 and U_3 are as defined in parts (a) and (c) above. Identify the distribution of U_4 by name, taking care to include any/all relevant parameter(s).

4. Let $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim} f(y)$ where

$$f(y) = e^{-(y-\theta)} \cdot \mathbb{1}_{\{y \geq \theta\}}$$

   (a) Find $f_{Y_{(1)}}(y)$, the density of the first order statistic.

(b) Is $Y_{(1)}$ an unbiased estimator for θ? Is it asymptotically unbiased?

(c) Compute MSE$(Y_{(1)}, \theta)$.

(d) Use the definition of consistency to show that $Y_{(1)}$ is a consistent estimator for θ.

5. (**7.52 from the Textbook**) Resistors to be used in a circuit have average resistance 200 ohms and standard deviation 10 ohms. Suppose 25 of these resistors are randomly selected to be used in a circuit.

   (a) What is the probability that the average resistance for the 25 resistors is between 199 and 202 ohms?

   (b) Find the probability that the total resistance does not exceed 5100 ohms.

6. (**9.6 from the 120A Textbook**) Nate is a competitive eater specializing in eating hot dogs. From past experience we know that it takes him on average 15 seconds to consume one hot dog, with a standard deviation of 4 seconds. In this year's hot dog eating contest he hopes to consume 64 hot dogs in just 15 minutes. Use the CLT to approximate the probability that he achieves this feat of skill.

7. (**7.42 from the Textbook**) The fracture strength of tempered glass averages 14 (measured in thousands of pounds per square inch) and has standard deviation 2.

   (a) What is the probability that the average fracture strength of 100 randomly selected pieces of this glass exceeds 14.5?

   (b) Find an interval that includes, with probability 0.95, the average fracture strength of 100 randomly selected pieces of this glass.

8. Suppose $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim}$ Unif$[-\theta, \theta]$ for some $\theta > 0$.

   (a) Find $\widehat{\theta}_{\text{MM}}$, the Method of Moments estimator for θ.

   (b) ~~Is $\widehat{\theta}_{\text{MM}}$ an unbiased estimator for θ? Is it asymptotically unbiased?~~

   (c) Is $\widehat{\theta}_{\text{MM}}$ a consistent estimator for θ?

9. Let $\{Y_i\}_{i=1}^n$ denote an i.i.d. sample from a distribution with density $f(y)$. Find $\widehat{\theta}_{\text{MM}}$, the method of moments estimator for θ, where:

   (a) $f(y) = (\theta + 1)y^{\theta} \cdot \mathbb{1}_{\{y \in [0,1]\}}$, and $\theta > -1$.

   (b) $f(y) = \left(\frac{2}{\theta^2}\right)(\theta - y) \cdot \mathbb{1}_{\{0 \le y \le \theta\}}$, and $\theta > 0$.