QUIZ 02

PSTAT 120B: Mathematical Statistics, I
Summer Session A, 2024 with Instructor: Ethan P. Marzban

Your Information

Your Name: ______________________________
(*First and Last*)

Your NetID: ______________________________
(*NOT Perm Number*)

Your Section: **2pm (Hyuk-Jean)** **3pm (Hyuk-Jean)** **4pm (Minwoo)** **5pm (Minwoo)**
(*Circle One*)

Instructions

- **PLEASE DO NOT REMOVE ANY PAGES FROM THIS QUIZ.**
- You have **20 minutes** to complete this quiz.
- The use of **calculators** is permitted, but the use of any other aids (including, but not limited to, notes, phones, laptops, etc.) is prohibited.
- Ensure you show **all your work**; correct answers with no supporting justification will not receive full credit.
- Kindly note that any instances of academic misconduct (including, but not limited to: unauthorized collaboration, or the use of unauthorized resources) will be dealt with in the strictest manner possible.
- Please refrain from discussing the contents of this quiz with **ANYONE**, until given the all-clear by the Instructor.
- **Good Luck!**

This quiz contains 15 points in total

The Quiz Begins on the Next Page

Problem 1. Let $Y_1, Y_2 \overset{\text{i.i.d.}}{\sim}$ Unif$[0,1]$, and define $U := Y_1 + Y_2$. In this problem, we will work toward deriving $f_U(u)$, the density of U, using the CDF Method.

(a) (3 points) For a fixed $u \in [0,1]$, show that $F_U(u)$, the CDF of U at the point u, is given by

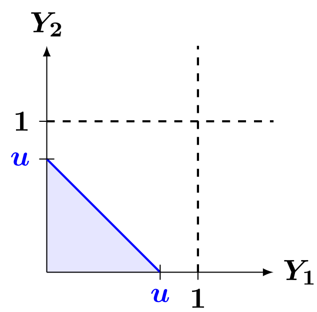
$$F_U(u) = \frac{1}{2}u^2$$

Be sure to show all your steps and justify your logic - failure to do so may incur point penalties!

Solution: By definition of the CDF, we have

$$F_U(u) := \mathbb{P}(U \leq u) = \mathbb{P}(Y_1 + Y_2 \leq u) = \mathbb{P}(Y_2 \leq u - Y_1)$$

This can be computed as a double integral over the following region:



Here's a little more detail on how we knew to draw the region above: the curve $y_2 = u - y_1$ is a line with slope -1 and intercept u, hence the thick blue line in the picture above. Additionally, since we are told (in this part) to assume $u \in [0,1]$, we knew to set the vertical and horizontal intercepts of the blue line to be less than 1.

Now, because our joint density is simply 1 over its support,

$$F_U(u) = \iint_{\mathcal{R}} (1)\ \mathrm{d}A = \text{Area}(\mathcal{R})$$

where $\mathcal{R}$ is the blue triangle sketched above. The desired area can be found using the formula for the area of a triangle: this triangle has base and height both equal to u, and so its area is simply

$$\text{Area} = \frac{1}{2}(u) \cdot (u) = \frac{1}{2}u^2$$

which is the desired result.

(b) (3 points) For a fixed $u \in [1,2]$, show that $F_U(u)$, the CDF of U at the point u, is given by

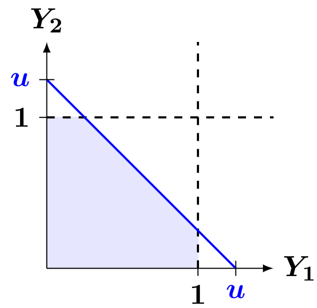
$$F_U(u) = 1 - \frac{1}{2}(2-u)^2$$

Be sure to show all your steps and justify your logic - failure to do so may incur point penalties!

Solution: Again, by definition of the CDF, we have

$$F_U(u) := \mathbb{P}(U \leq u) = \mathbb{P}(Y_1 + Y_2 \leq u) = \mathbb{P}(Y_2 \leq u - Y_1)$$

This can be computed as a double integral over the following region:



The way we sketch this region is much like the way we sketched the region in part (a); now, however, since we are told to assume $u \in [1, 2]$, we knew to set the vertical and horizontal intercepts of the blue line to a point that is *greater* than 1.

Again, we can utilize the joint uniformity of Y_1 and Y_2 to find the desired probability simply as an area. Rather than trying to find the area of the blue trapezoid, it will be easier to do 1 minus the area of the triangle in the upper-right corner. This triangle has base and height equal to

$$1 - (u - 1) = 2 - u$$

and so the area of the triangle is

$$\frac{1}{2}(2-u)(2-u) = \frac{1}{2}(2-u)^2$$

and the area of the trapezoid is simply

$$1 - \frac{1}{2}(2-u)^2$$

(c) (3 points) Combining your answers to the above parts, derive an expression for $f_U(u)$, the probability density function of U. Be sure to include the support of U in your final answer!

Solution: What we have from the results provided in the statements of parts (a) and (b) is that

$$F_U(u) = \begin{cases} \frac{1}{2}u^2 & \text{if } 1 \leq u < 2 \\ 1 - \frac{1}{2}(2-u)^2 & \text{if } 1 \leq u < 2 \end{cases}$$

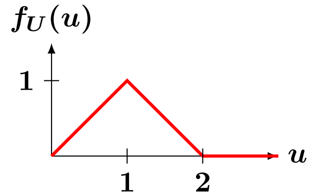
Additionally, since $S_{Y_1} = [0, 1]$ and $S_{Y_2} = [0, 1]$, we see that $S_U = [0, 2]$, meaning we can complete the CDF as

$$F_U(u) = \begin{cases} 0 & \text{if } u < 0 \\ \frac{1}{2}u^2 & \text{if } 1 \leq u < 2 \\ 1 - \frac{1}{2}(2-u)^2 & \text{if } 1 \leq u < 2 \\ 1 & \text{if } u > 0 \end{cases}$$

Taking the derivative gives

$$f_U(u) = \begin{cases} u & \text{if } 0 \le u < 1 \\ 2-u & \text{if } 1 \le u < 2 \\ 0 & \text{otherwise} \end{cases}$$

As an Aside (Not Required for Full Points): This density looks like



The distribution is U is a special case of what is known as the **triangular distribution**. BTW, this entire quiz question was actually a lecture exercise I encouraged everyone to work on on their own - take a look at slide 14 (page 16) of the Topic 2.5 slides!

Problem 2. Let $Y_1, Y_2, Y_3 \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 4)$; i.e. Var$(Y_i) = 4$ for $i = 1, 2, 3$.

(a) (3 points) What is the distribution of $U_1 := (Y_1^2 + Y_2^2 + Y_3^2)/12$? Be sure to include the distribution's name, as well as any/all relevant parameter(s).

Solution: Note that

$$\frac{Y_1^2 + Y_2^2 + Y_3^2}{4} = \left(\frac{Y_1}{2}\right)^2 + \left(\frac{Y_2}{2}\right)^2 + \left(\frac{Y_3}{2}\right)^2 \sim \chi_3^2 \sim \text{Gamma}(3/2, 2)$$

Therefore, since if $Y \sim$ Gamma(α, β) then $(cY) \sim$ Gamma$(\alpha, c\beta)$ for a positive constant c, we have that

$$U_3 = \frac{1}{3}\left[\left(\frac{Y_1}{2}\right)^2 + \left(\frac{Y_2}{2}\right)^2 + \left(\frac{Y_3}{2}\right)^2\right] \sim \text{Gamma}(3/2, 2/3)$$

PLEASE NOTE: Admittedly, I didn't explicitly emphasize this scaling result of Gamma distributions much - so, after some consideration, **I've decided to give everyone full points for 2(a)**. You should, however, be familiar with the scaling property of Gamma distributions for Midterm 2.

(b) (3 points) What is the distribution of

$$U_2 := \sqrt{2}\left(\frac{\frac{1}{2}(Y_1 + Y_2)}{\sqrt{\left[Y_1 - \frac{Y_1+Y_2}{2}\right]^2 + \left[Y_2 - \frac{Y_1+Y_2}{2}\right]^2}}\right)$$

Be sure to include the distribution's name, as well as any/all relevant parameter(s). **Hint:** You do not need to use any transformation method, nor do you need to perform any integration. Instead, try and recognize the quantity U_2 as something whose distribution was given in lecture.

Solution: Note:

$$\begin{aligned}
\sqrt{2}\left(\frac{\frac{1}{2}(Y_1+Y_2)}{\sqrt{\left[Y_1-\frac{Y_1+Y_2}{2}\right]^2+\left[Y_2-\frac{Y_1+Y_2}{2}\right]^2}}\right) &= \sqrt{2}\left(\frac{\overline{Y}_2}{\sqrt{\frac{1}{2-1}\sum_{i=1}^{2}(Y_i-\overline{Y}_2)^2}}\right) \\
&= \sqrt{2}\left(\frac{\overline{Y}_2-0}{S_2}\right) \sim t_{2-1} \sim t_1
\end{aligned}$$

END OF QUIZ

Selected Discrete Distributions

Distribution	PMF	Expec./Var.	MGF
$X \sim \mathsf{Bin}(n,p)$	$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$, $k \in \{0,\cdots,n\}$	$\mathbb{E}[X] = np$ $\mathsf{Var}(X) = np(1-p)$	$M_X(t) = (1-p+pe^t)^n$
$X \sim \mathsf{Geom}(p)$	$p_X(k) = p(1-p)^{k-1}$, $k \in \{1,2,\cdots\}$	$\mathbb{E}[X] = \frac{1}{p}$ $\mathsf{Var}(X) = \frac{1-p}{p^2}$	$M_X(t) = \frac{pe^t}{1-(1-p)e^t}$, $t < -\ln(1-p)$
$X \sim \mathsf{Geom}(p)$	$p_X(k) = p(1-p)^{k}$, $k \in \{0,1,\cdots\}$	$\mathbb{E}[X] = \frac{1-p}{p}$ $\mathsf{Var}(X) = \frac{1-p}{p^2}$	$M_X(t) = \frac{p}{1-(1-p)e^t}$, $t < -\ln(1-p)$
$X \sim \mathsf{NegBin}(r,p)$	$p_X(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$, $k \in \{r, r+1,\cdots\}$	$\mathbb{E}[X] = \frac{r}{p}$ $\mathsf{Var}(X) = \frac{r(1-p)}{p^2}$	$M_X(t) = \left[\frac{pe^t}{1-(1-p)e^t}\right]^r$, $t < -\ln(1-p)$
$X \sim \mathsf{Pois}(\lambda)$	$p_X(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$, $k \in \{0,1,\cdots\}$	$\mathbb{E}[X] = \lambda$ $\mathsf{Var}(X) = \lambda$	$M_X(t) = \exp\left\{\lambda(e^t-1)\right\}$

Selected Continuous Distributions

Distribution	PDF	Expec./Var.	MGF
$X \sim \mathsf{Unif}[a,b]$	$f_X(x) = \dfrac{1}{b-a} \cdot \mathbb{1}_{\{x \in [a,b]\}}$	$\mathbb{E}[X] = \frac{a+b}{2}$ $\mathsf{Var}(X) = \frac{(b-a)^2}{12}$	$M_X(t) = \begin{cases} \frac{e^{tb}-e^{ta}}{t(b-a)} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$
$X \sim \mathsf{Exp}(\beta)$	$f_X(x) = \dfrac{1}{\beta} e^{-x/\beta} \cdot \mathbb{1}_{\{x \geq 0\}}$	$\mathbb{E}[X] = \beta$ $\mathsf{Var}(X) = \beta^2$	$M_X(t) = (1-\beta t)^{-1},\ t < 1/\beta$
$X \sim \mathsf{Gamma}(\alpha,\beta)$	$f_X(x) = \dfrac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \cdot \mathbb{1}_{\{x \geq 0\}}$	$\mathbb{E}[X] = \alpha\beta$ $\mathsf{Var}(X) = \alpha\beta^2$	$M_X(t) = (1-\beta t)^{-\alpha},\ t < 1/\beta$
$X \sim \mathcal{N}(\mu,\sigma^2)$	$f_X(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$	$\mathbb{E}[X] = \mu$ $\mathsf{Var}(X) = \sigma^2$	$M_X(t) = \exp\left\{\mu t + \frac{\sigma^2}{2} t\right\}$

$$\frac{\mathsf{d}}{\mathsf{d}x}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \qquad \int_y^\infty \frac{1}{\theta} e^{-x/\theta}\,\mathsf{d}x = e^{-y/\theta} \qquad \Gamma(r) := \int_0^\infty t^{r-1} e^{-t}\,\mathsf{d}t$$

- If $Y \sim \mathcal{N}(0,1)$, then $Y^2 \sim \chi_1^2$.
- If $Y \sim \mathcal{N}(\mu,\sigma^2)$, then $U := [(Y-\mu)/\sigma] \sim \mathcal{N}(0,1)$
- Given independent random variables $\{Y_i\}_{i=1}^n$ with $Y_i \sim \chi^2_{\nu_i}$, then $(Y_1 + \cdots + Y_n) \sim \chi^2_{\nu_1+\cdots+\nu_n}$
- Given $Y_1, Y_2, \cdots \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu,\sigma^2)$, $\overline{Y}_n := \frac{1}{n}\left(\sum_{i=1}^n Y_i\right)$ and $S_n := \sqrt{\frac{1}{n-1}\sum_{i=1}^n (Y_i - \overline{Y}_n)^2}$,

$$\sqrt{n}\left(\frac{\overline{Y}_n - \mu}{\sigma}\right) \sim \mathcal{N}(0,1); \qquad \sqrt{n}\left(\frac{\overline{Y}_n - \mu}{S_n}\right) \sim t_{n-1}; \qquad \left(\frac{n-1}{\sigma^2}\right) S_n^2 \sim \chi^2_{n-1}$$

- $f_{Y_{(1)}} = n[1 - F_Y(y)]^{n-1} f_Y(y);$ $\qquad f_{Y_{(n)}}(y) = n[F_Y(y)]^{n-1} f_Y(y)$