DISCUSSION WORKSHEET 01

PSTAT 120B: Mathematical Statistics, I
Summer Session A, 2024 with Instructor: Ethan P. Marzban

Welcome to our first PSTAT 120B Discussion Section! Discussion worksheets are designed to give you additional practice with material covered in lecture.

Conceptual Review

(a) What is a **conditional mass/density function**? For what values is it defined?

(b) What is a **conditional expectation**?

(c) ~~What are the **Law of Iterated Expectations** and **Law of Total Variance**~~?

(d) What is the **Gamma distribution**? Specifically, what is its density function? What are its expectation and variance?

Problem 1: Conditional Distributions/Expectations

Let (X, Y) be a bivariate random vector with joint probability density function (p.d.f.) given by

$$f_{X,Y}(x, y) = \begin{cases} \lambda y e^{-y(x+\lambda)} & \text{if } x \geq 0,\ y \geq 0 \\ 0 & \textbf{otherwise} \end{cases}$$

where $\lambda > 0$ is a fixed constant.

a) Find $f_Y(y)$, the marginal density of Y and use this to identify Y as belonging to a known distribution. **Be sure to include any/all relevant parameter(s)!**

b) Find $f_{X|Y}(x \mid y)$, the density of $(X \mid Y = y)$, and use this to identify $(X \mid Y = y)$ as belonging to a known distribution. **Be sure to include any/all relevant parameter(s)!**

c) ~~Set up **but do not evaluate** and integral corresponding to $\mathbb{E}[X]$, that only involves the marginal density function of Y.~~

Hint: Law of Iterated Expectations

Solution:

a) To find the marginal density of Y, we integrate x out of the joint:

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} \lambda y e^{-y(x+\lambda)} \cdot \mathbb{1}_{\{x \geq 0\}} \cdot \mathbb{1}_{\{y \geq 0\}} \, \mathrm{d}x \\ &= \lambda y e^{-y\lambda} \cdot \mathbb{1}_{\{y \geq 0\}} \cdot \int_0^{\infty} e^{-xy} \, \mathrm{d}x \\ &= \lambda \cancel{y} e^{-y\lambda} \cdot \mathbb{1}_{\{y \geq 0\}} \cdot \frac{1}{\cancel{y}} = \boxed{\lambda e^{-\lambda y} \cdot \mathbb{1}_{\{y \geq 0\}}} \end{aligned}$$

which gives us $Y \sim \text{Exp}(1/\lambda)$ (recall that, in PSTAT 120B, we use the *mean* of the Exponential distribution as its parameter).

b) We first note that $f_{X|Y}(x \mid y)$ is defined only for y such that $f_Y(y) \geq 0$. From our answer to part (a), we know that $f_Y(y) \geq 0$ only when $y \geq 0$; hence, let us *a priori* set $y > 0$, and compute

$$\begin{aligned} f_{X|Y}(x \mid y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \frac{\cancel{\lambda} y e^{-y(x+\lambda)} \cdot \mathbb{1}_{\{x \geq 0\}} \cdot \cancel{\mathbb{1}_{\{y \geq 0\}}}}{\cancel{\lambda} e^{-\lambda y} \cdot \cancel{\mathbb{1}_{\{y \geq 0\}}}} \\ &= \frac{y e^{-xy} \cancel{e^{-\lambda y}}}{\cancel{e^{-\lambda y}}} \cdot \mathbb{1}_{\{x \geq 0\}} = y e^{-xy} \cdot \mathbb{1}_{\{x \geq 0\}} \end{aligned}$$

and hence, $(X \mid Y = y) \sim \text{Exp}(1/y)$.

c) **THIS PART WAS NOT COVERED**.

Problem 2: The Gamma Distribution

Recall (from lecture) that if $X \sim \text{Gamma}(\alpha, \beta)$, then X has density given by

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \cdot \mathbb{1}_{\{x \geq 0\}}$$

where $\Gamma(\alpha)$ denotes the **gamma function**, defined as

$$\Gamma(r) := \int_0^\infty x^{r-1} e^{-x} \, \mathrm{d}x \text{if } r \geq 0$$

and $\Gamma(0) := 1$.

a) Show that X has MGF (moment generating function) given by

$$M_X(t) = \begin{cases} (1 - \beta t)^{-\alpha} & \text{if } t < 1/\beta \\ \infty & \text{otherwise} \end{cases}$$

b) Let $Y \sim \chi^2_\nu$. Use your answer to part (a) to derive the MGF $M_Y(t)$ of Y.

c) What is another name for the χ^2_2 distribution? Be sure to give the distribution's name and also list out any/all relevant parameter(s)!

Solution:

a) $$\begin{aligned} M_X(t) := \mathbb{E}[e^{tX}] &= \int_{-\infty}^\infty e^{tx} f_X(x) \, \mathrm{d}x \\ &= \int_{-\infty}^\infty e^{tx} \cdot \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \cdot \mathbb{1}_{\{x \geq 0\}} \, \mathrm{d}x \\ &= \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot \int_0^\infty x^{\alpha-1} e^{-x\left(\frac{1}{\beta} - t\right)} \, \mathrm{d}x \end{aligned}$$

Now, note that this integral will diverge whenever the exponent is negative. Hence, the MGF is finite only when

$$\frac{1}{\beta} - t > 0 \iff t < 1/\beta$$

Hence, let us fix a $t < 1/\beta$, and proceed.

$$\begin{aligned} M_X(t) &= \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot \int_0^\infty x^{\alpha-1} e^{-x\left(\frac{1}{\beta}-t\right)} \, \mathrm{d}x \\ &= \frac{1}{\beta^\alpha} \cdot \frac{1}{\left(\frac{1}{\beta} - t\right)^\alpha} \cdot \int_0^\infty \frac{1}{\Gamma(\alpha)\left[\frac{1}{\left(\frac{1}{\beta}-t\right)}\right]^\alpha} \cdot x^{\alpha-1} e^{-x\left(\frac{1}{\beta}-t\right)} \, \mathrm{d}x \end{aligned}$$

The integrand is now the density of a Gamma$(\alpha, [1/\beta] - t)$ distribution; since we are integrating this density over its entire support, the entire integral must be unity. Hence, for $t < 1/\beta$

$$\begin{aligned} M_X(t) &= \frac{1}{\beta^\alpha} \cdot \frac{1}{\left(\frac{1}{\beta} - t\right)^\alpha} \\ &= \frac{1}{\beta^\alpha} \cdot \frac{1}{\left(\frac{1-\beta t}{\beta}\right)^\alpha} = \frac{1}{(1-\beta t)^\alpha} = (1-\beta t)^{-\alpha} \end{aligned}$$

which recovers the desired result.

There is another way to solve the integral: instead of relating the integrand to a Gamma *density*, we can relate the entire integral to a Gamma *function*. Specifically, return to this step:

$$M_X(t) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot \int_0^\infty x^{\alpha-1} e^{-x\left(\frac{1}{\beta}-t\right)} \, \mathrm{d}x$$

Let's make a $u-$substitution:

$$u = x\left(\frac{1}{\beta} - t\right) = x \cdot \left(\frac{1-\beta t}{\beta}\right)$$

This gives

$$x = \left(\frac{\beta}{1-\beta t}\right) u$$

and, consequently, $\mathrm{d}x = \left(\frac{\beta}{1-\beta t}\right) \, \mathrm{d}u$ Hence, we obtain

$$\begin{aligned} M_X(t) &= \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot \int_0^\infty x^{\alpha-1} e^{-x\left(\frac{1}{\beta}-t\right)} \, \mathrm{d}x \\ &= \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot \int_0^\infty \left[\left(\frac{\beta}{1-\beta t}\right) u\right]^{\alpha-1} e^{-u} \left(\frac{\beta}{1-\beta t}\right) \, \mathrm{d}u \\ &= \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot \left(\frac{\beta}{1-\beta t}\right)^{\alpha-1} \cdot \left(\frac{\beta}{1-\beta t}\right) \cdot \int_0^\infty u^{\alpha-1} e^{-u} \, \mathrm{d}u \end{aligned}$$

The integral is now precisely the definition of $\Gamma(\alpha)$; hence

$$\begin{aligned} M_X(t) &= \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot \left(\frac{\beta}{1 - \beta t} \right)^{\alpha - 1} \cdot \left(\frac{\beta}{1 - \beta t} \right) \cdot \int_0^\infty u^{\alpha - 1} e^{-u} \, \mathrm{d}u \\ &= \frac{1}{\cancel{\Gamma(\alpha)}\,\cancel{\beta^\alpha}} \cdot \left(\frac{\cancel{\beta}}{1 - \beta t} \right)^{\alpha} \cdot \cancel{\Gamma(\alpha)} \\ &= \left(\frac{1}{1 - \beta t} \right)^{\alpha} = (1 - \beta t)^{-\alpha} \end{aligned}$$

just as we had before.

b) Recall that the χ^2_ν distribution is equivalent to the Gamma$(\nu/2, 2)$ distribution. Hence, we only need to plug $\alpha = \nu/2$ and $\beta = 2$ into our MGF from part (a):

$$M_Y(t) = \begin{cases} (1 - 2t)^{-\nu/2} & \text{if } t < 1/2 \\ \infty & \text{otherwise} \end{cases}$$

c) Again, the χ^2_ν distribution is equivalent to the Gamma$(\nu/2, 2)$ distribution. Hence, the χ^2_2 distribution is equivalent to the Gamma$(1, 2)$ distribution which is itself equivalent to the Exp(2) distribution.