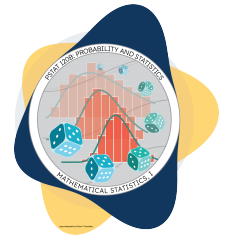


DISCUSSION WORKSHEET 04

PSTAT 120B: Mathematical Statistics, I
Summer Session A, 2024 with Instructor: Ethan P. Marzban



Conceptual Review

- What are the main goals of **inferential statistics**?
- What is a **population**? What is a **sample**? What is the difference between a sample and an **observed instance** (aka **realization**) of a sample?
- What is a **statistic**? When does a statistic become an **estimator**?
- What is the notion of a **sampling distribution**? How does that tie into the notions of **bias** and **unbiasedness**?
- What is the **mean square error** of an estimator?

Problem 1: Exercise 8.12 from the Textbook

The reading on a voltage meter connected to a test circuit is uniformly distributed over the interval $(\theta, \theta + 1)$, where θ is the true but unknown voltage of the circuit. Suppose that Y_1, Y_2, \dots, Y_n denote a random sample of such readings.

- Show that \bar{Y}_n is a biased estimator of θ and compute the bias.

Solution: We have seen (several times) that the expected value of the sample mean is the population mean. Hence,

$$\mathbb{E}[\bar{Y}_n] = \mathbb{E}[Y_1] = \frac{\theta + \theta + 1}{2} = \frac{2\theta + 1}{2} = \theta + \frac{1}{2}$$

Since this is (in general) not equal to θ , we have shown that \bar{Y}_n is a biased estimator of θ . Additionally, by the definition of bias,

$$\text{Bias}(\bar{Y}_n, \theta) := \mathbb{E}[\bar{Y}_n] - \theta = \theta + \frac{1}{2} - \theta = \frac{1}{2}$$

- Find a function of \bar{Y}_n that is an unbiased estimator of θ .

Solution: Since \bar{Y}_n (on average) overestimates θ by a constant factor of $1/2$, it seems plausible to construct an unbiased estimator for θ by taking \bar{Y}_n and subtracting $1/2$. That is, define the following estimator for θ :

$$\hat{\theta}_n := \bar{Y}_n - \frac{1}{2}$$

Then,

$$\mathbb{E}[\hat{\theta}_n] = \mathbb{E}\left[\bar{Y}_n - \frac{1}{2}\right] = \mathbb{E}[\bar{Y}_n] - \frac{1}{2} = \theta + \frac{1}{2} - \frac{1}{2} = \theta$$

thereby showing that $\hat{\theta}_n := \bar{Y}_n - \frac{1}{2}$ is an unbiased estimator of θ .

(c) Find $\text{MSE}(\bar{Y}_n, \theta)$.

Solution: By the Bias-Variance Decomposition,

$$\text{MSE}(\bar{Y}_n, \theta) = [\text{Bias}(\bar{Y}_n, \theta)]^2 + \text{Var}(\bar{Y}_n)$$

We computed $\text{Bias}(\bar{Y}_n, \theta)$ in part (a); additionally, we know that

$$\text{Var}(\bar{Y}_n) = \frac{\text{Var}(Y_1)}{n} = \frac{1}{12n}(\theta + 1 - \theta)^2 = \frac{1}{12n}$$

Therefore, putting everything together,

$$\begin{aligned} \text{MSE}(\bar{Y}_n, \theta) &= [\text{Bias}(\bar{Y}_n, \theta)]^2 + \text{Var}(\bar{Y}_n) \\ &= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{12n}\right) = \frac{3n+1}{12n} \end{aligned}$$