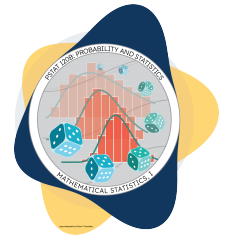


DISCUSSION WORKSHEET 05

PSTAT 120B: Mathematical Statistics, I
Summer Session A, 2024 with Instructor: Ethan P. Marzban



Conceptual Review

- What is a **population moment**? What is a **sample moment**?
- What is the **Method of Moments**? Does the Method of Moments produce consistent estimators? What about unbiased estimators?
- What is a **likelihood**?

Problem 1

Let $Y_1, \dots, Y_n \stackrel{\text{i.i.d.}}{\sim} f(y; \alpha)$ where

$$f(y; \alpha) = \frac{1 + \alpha y}{2} \cdot \mathbb{1}_{\{-1 \leq y \leq 1\}}$$

where $\alpha \in [-1, 1]$ is an unknown parameter.

- Find $\hat{\alpha}_{\text{MM}}$, the method of moments estimator for α .

Solution: We first compute

$$\mathbb{E}[Y_1] = \int_{-1}^1 y \cdot \frac{1 + \alpha y}{2} dy = \frac{\alpha}{3}$$

Therefore, $\hat{\alpha}_{\text{MM}}$ must satisfy

$$\frac{\hat{\alpha}_{\text{MM}}}{3} = \bar{Y}_n \iff \hat{\alpha}_{\text{MM}} = 3\bar{Y}_n$$

- Show that $\hat{\alpha}_{\text{MM}}$ is a consistent estimator for α .

Solution:

- By the WLLN, $\bar{Y}_n \xrightarrow{\mathbb{P}} \mathbb{E}[Y_1] = \frac{\alpha}{3}$
- By the CMT with $g(x) = 3x$, we have $3\bar{Y}_n \xrightarrow{\mathbb{P}} 3\mathbb{E}[Y_1] = e \cdot \frac{\alpha}{3} = \alpha \checkmark$

- Suppose we obtain the following observed instance of a sample from the above-stated distribution:

$$\vec{y} = (-0.38, 0.61, -0.13, 0.79, 0.89, -0.90, 0.11, 0.80)$$

What is an appropriate estimate for α based on this sample, using the method of moments?

Solution: We simply need to plug into our estimator to obtain an estimate:

$$\bar{y} = \frac{1}{8}(-0.38 + 0.61 - 0.13 + \dots + 0.80) = 0.22375$$

$$3\bar{y} = 3 \cdot 0.22375 \approx 0.67125$$

- (d) What is the approximate sampling distribution of $\hat{\alpha}_{MM}$, assuming a large sample size?

Solution: Assuming a large sample size, the CLT tells us that

$$\frac{\sqrt{n}(\bar{Y}_n - \mu)}{\sigma} \stackrel{d}{\approx} \mathcal{N}(0, 1)$$

Another way to say this is, for large n ,

$$\bar{Y}_n \stackrel{d}{\approx} \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

meaning

$$3\bar{Y}_n \stackrel{d}{\approx} \mathcal{N}\left(3\mu, \frac{9\sigma^2}{n}\right)$$

We already computed $\mu = \alpha/3$ in part (a); now all we need is to compute σ^2 :

$$\mathbb{E}[Y_i^2] = \int_{-1}^1 y^2 \cdot \frac{1 + \alpha y}{2} dy = \frac{1}{3}$$

$$\text{Var}(Y_i) = \mathbb{E}[Y_i^2] - (\mathbb{E}[Y_i])^2 = \frac{1}{3} - \left(\frac{\alpha}{3}\right)^2 = \frac{3 - \alpha^2}{9}$$

Hence,

$$\hat{\alpha}_{MM} = 3\bar{Y}_n \stackrel{d}{\approx} \mathcal{N}\left(\alpha, \frac{3 - \alpha^2}{n}\right)$$

- (e) Use your answer to part (d) to approximate the probability that $\hat{\alpha}_{MM}$ exceeds 1. Why is this a problem (i.e. why would we want this probability to be as small as possible)?

Solution:

$$\mathbb{P}(\hat{\alpha}_{MM} > 1) = \mathbb{P}\left(\frac{\hat{\alpha}_{MM} - \alpha}{\sqrt{\frac{3 - \alpha^2}{n}}} > \frac{1 - \alpha}{\sqrt{\frac{3 - \alpha^2}{n}}}\right)$$

$$\approx 1 - \Phi\left(\frac{1 - \alpha}{\sqrt{\frac{3 - \alpha^2}{n}}}\right) = 1 - \Phi\left(\sqrt{n} \cdot \frac{1 - \alpha}{\sqrt{3 - \alpha^2}}\right)$$

The problem explicitly tells us that α is constrained to be between -1 and 1 (indeed, otherwise

the density isn't even a valid probability density!). Hence, obtaining an estimate above 1 is problematic. However, note that as $n \rightarrow \infty$ this probability goes to 0, so in the long run we should be okay! (Also, keep in mind that the CLT provides an *approximation* to the sampling distribution of $\hat{\alpha}_{MM}$.)