DISCUSSION WORKSHEET 05

PSTAT 120B: Mathematical Statistics, I
Summer Session A, 2024 with Instructor: Ethan P. Marzban

Conceptual Review

(a) What is a **population moment**? What is a **sample moment**?

(b) What is the **Method of Moments**? Does the Method of Moments produce consistent estimators? What about unbiased estimators?

(c) What is a **likelihood**?

Problem 1

Let $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim} f(y; \alpha)$ where

$$f(y; \alpha) = \frac{1 + \alpha y}{2} \cdot \mathbb{1}_{\{-1 \leq y \leq 1\}}$$

where $\alpha \in [-1, 1]$ is an unknown parameter.

(a) Find $\widehat{\alpha}_{\mathrm{MM}}$, the method of moments estimator for α.

Solution: We first compute

$$\mathbb{E}[Y_1] = \int_{-1}^{1} y \cdot \frac{1 + \alpha y}{2} \, \mathrm{d}y = \frac{\alpha}{3}$$

Therefore, $\widehat{\alpha}_{\mathrm{MM}}$ must satisfy

$$\frac{\widehat{\alpha}_{\mathrm{MM}}}{3} = \overline{Y}_n \iff \widehat{\alpha}_{\mathrm{MM}} = 3\overline{Y}_n$$

(b) Show that $\widehat{\alpha}_{\mathrm{MM}}$ is a consistent estimator for α.

Solution:

- By the WLLN, $\overline{Y}_n \xrightarrow{\mathbb{P}} \mathbb{E}[Y_1] = \frac{\alpha}{3}$
- By the CMT with $g(x) = 3x$, we have $3\overline{Y}_n \xrightarrow{\mathbb{P}} 3\mathbb{E}[Y_1] = e \cdot \frac{\alpha}{3} = \alpha$ ✓

(c) Suppose we obtain the following observed instance of a sample from the above-stated distribution:

$$\vec{\boldsymbol{y}} = (-0.38,\ 0.61,\ -0.13,\ 0.79,\ 0.89,\ -0.90,\ 0.11,\ 0.80)$$

What is an appropriate estimate for α based on this sample, using the method of moments?

Solution: We simply need to plug into our estimator to obtain an estimate:

$$\overline{y} = \frac{1}{8}(-0.38 + 0.61 - 0.13 + \cdots + 0.80) = 0.22375$$
$$3\overline{y} = 3 \cdot 0.22375 \approx 0.67125$$

(d) What is the approximate sampling distribution of $\widehat{\alpha}_{\text{MM}}$, assuming a large sample size?

Solution: Assuming a large sample size, the CLT tells us that

$$\frac{\sqrt{n}(\overline{Y}_n - \mu)}{\sigma} \overset{d}{\approx} \mathcal{N}(0, 1)$$

Another way to say this is, for large n,

$$\overline{Y}_n \overset{d}{\approx} \mathcal{N}\left(\mu\,,\ \frac{\sigma^2}{n}\right)$$

meaning

$$3\overline{Y}_n \overset{d}{\approx} \mathcal{N}\left(3\mu\,,\ \frac{9\sigma^2}{n}\right)$$

We already computed $\mu = \alpha/3$ in part (a); now all we need is to compute σ^2:

$$\mathbb{E}[Y_i^2] = \int_{-1}^{1} y^2 \cdot \frac{1 + \alpha y}{2}\,\mathrm{d}y\,\mathrm{d}y = \frac{1}{3}$$
$$\mathrm{Var}(Y_i) = \mathbb{E}[Y_i^2] - (\mathbb{E}[Y_i])^2 = \frac{1}{3} - \left(\frac{\alpha}{3}\right)^2 = \frac{3 - \alpha^2}{9}$$

Hence,

$$\widehat{\alpha}_{\text{MM}} = 3\overline{Y}_n \overset{d}{\approx} \mathcal{N}\left(\alpha,\ \frac{3 - \alpha^2}{n}\right)$$

(e) Use your answer to part (d) to approximate the probability that $\widehat{\alpha}_{\text{MM}}$ exceeds 1. Why is this a problem (i.e. why would we want this probability to be as small as possible)?

Solution:

$$\mathbb{P}(\widehat{\alpha}_{\text{MM}} > 1) = \mathbb{P}\left(\frac{\widehat{\alpha}_{\text{MM}} - \alpha}{\sqrt{\frac{3-\alpha^2}{n}}} > \frac{1 - \alpha}{\sqrt{\frac{3-\alpha^2}{n}}}\right)$$
$$\approx 1 - \Phi\left(\frac{1-\alpha}{\sqrt{\frac{3-\alpha^2}{n}}}\right) = 1 - \Phi\left(\sqrt{n} \cdot \frac{1 - \alpha}{\sqrt{3 - \alpha^2}}\right)$$

The problem explicitly tells us that α is constrained to be between -1 and 1 (indeed, otherwise the density isn't even a valid probability density!). Hence, obtaining an estimate above 1 is problematic. However, note that as $n \to \infty$ this probability goes to 0, so in the long run we should be okay! (Also, keep in mind that the CLT provides an *approximation* to the sampling distribution of $\widehat{\alpha}_{\mathsf{MM}}$.)