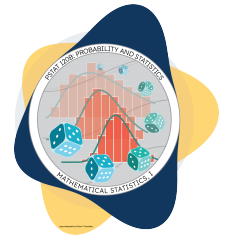


## DISCUSSION WORKSHEET 06

**PSTAT 120B: Mathematical Statistics, I**  
**Summer Session A, 2024** with Instructor: Ethan P. Marzban



### Conceptual Review

- What is a **likelihood**? What about a **log-likelihood**?
- How do we obtain a **maximum likelihood estimator** for a parameter? What do we do if the likelihood is nondifferentiable in the parameter of interest?
- What is a **sufficient statistic**? How can the **factorization theorem** help us find a statistic that is sufficient for a given parameter?

### Problem 1

Let  $Y_1, \dots, Y_n \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$ , where both  $\mu \in \mathbb{R}$  and  $\sigma^2 > 0$  are unknown parameters.

- Derive an expression for  $\mathcal{L}_{\vec{Y}}(\mu, \sigma^2)$ , the likelihood of the sample  $\vec{Y}$ . Recall that, since our sample is assumed to be i.i.d.,

$$\mathcal{L}_{\vec{Y}}(\mu, \sigma^2) = \prod_{i=1}^n f_{Y_i}(y_i; \mu, \sigma^2)$$

- Derive an expression for  $\ell_{\vec{Y}}(\mu, \sigma^2)$ , the log-likelihood of the sample  $\vec{Y}$ . Also compute

$$\frac{\partial}{\partial \mu} \ell_{\vec{Y}}(\mu, \sigma^2) \quad \text{and} \quad \frac{\partial}{\partial \sigma^2} \ell_{\vec{Y}}(\mu, \sigma^2)$$

- In the two derivatives you found in part (b), replace all instances of  $\mu$  with  $\hat{\mu}_{\text{MLE}}$ , all instances of  $\sigma^2$  with  $\hat{\sigma}_{\text{MLE}}^2$ . Set the resulting expressions equal to zero and solve for  $\hat{\mu}_{\text{MLE}}$  and  $\hat{\sigma}_{\text{MLE}}^2$ .
- The **equivariance** property of maximum likelihood estimators is as follows: given the MLE  $\hat{\theta}_{\text{MLE}}$  for a parameter  $\theta$ , the MLE of  $\tau(\theta)$  [where  $\tau(\cdot)$  is an appropriately-behaved function] is  $\tau(\hat{\theta}_{\text{MLE}})$ . For example, the MLE of  $\theta^3$  is  $(\hat{\theta}_{\text{MLE}})^3$ .

Use the equivariance property and your answer to part (c) to derive an expression for the maximum likelihood estimator of  $\sigma$ , the population *standard deviation*.

### Problem 2

Something's gone awry with *GachoPop*'s newest bottling machine! Specifically, the new soda dispenser doesn't fill each bottle entirely - rather, the proportion  $Y_i$  of a randomly-selected bottle that is full of soda follows the distribution with density

$$f(y; \theta) = \theta y^{\theta-1} \cdot \mathbb{1}_{\{0 \leq y \leq 1\}}$$

where  $\theta > 0$  is an unknown parameter. Let  $Y_1, \dots, Y_n$  denote the proportion of fill contained in  $n$  randomly-selected *GachoPop* bottles.

- Find  $\hat{\theta}_{\text{MLE}}$ , the maximum likelihood estimator for  $\theta$ .
- Company regulations requires that any bottles with fewer than 0.8 fill be labeled as "not fit for sale." Let  $\tau$  denote the true proportion of bottles that end up labeled as "not fit for sale" - find  $\hat{\tau}_{\text{MLE}}$ , the maximum likelihood estimator for  $\tau$ .

**Hint:** Use the equivariance property

### Problem 3

Let  $Y_1, \dots, Y_n \stackrel{\text{i.i.d.}}{\sim} \text{Unif}[\theta, 1]$  where  $\theta < 1$  is an unknown parameter.

- Find  $\hat{\theta}_{\text{MM}}$ , the method of moments estimator for  $\theta$ .
- Show that the likelihood of the sample  $\vec{Y}$  is given by

$$\mathcal{L}_{\vec{Y}}(\theta) = \left( \frac{1}{1-\theta} \right)^n \cdot \mathbb{1}_{\{\theta \leq Y_{(1)}\}} \cdot \mathbb{1}_{\{Y_{(n)} \leq 1\}}$$

where  $Y_{(1)}$  denotes the first order statistic (sample minimum) and  $Y_{(n)}$  denotes the  $n^{\text{th}}$  order statistic (sample maximum). Justify your logic.

- Find  $\hat{\theta}_{\text{MLE}}$ , the maximum likelihood estimator for  $\theta$ .
- Find the exact sampling distribution of  $\hat{\theta}_{\text{MLE}}$ , and use this to determine whether or not  $\hat{\theta}_{\text{MLE}}$  is an unbiased estimator for  $\theta$ .
- Show that  $U := Y_{(1)}$ , the first order statistic, is a sufficient sufficient statistic for  $\theta$ .