DISCUSSION WORKSHEET 06

PSTAT 120B: Mathematical Statistics, I
Summer Session A, 2024 with Instructor: Ethan P. Marzban

Conceptual Review

(a) What is a **likelihood**? What about a **log-likelihood**?

(b) How do we obtain a **maximum likelihood estimator** for a parameter? What do we do if the likelihood is nondifferentiable in the parameter of interest?

(c) What is a **sufficient statistic**? How can the **factorization theorem** help us find a statistic that is sufficient for a given parameter?

Problem 1

Let $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$, where both $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ are unknown parameters.

(a) Derive an expression for $\mathcal{L}_{\vec{\boldsymbol{Y}}}(\mu, \sigma^2)$, the likelihood of the sample $\vec{\boldsymbol{Y}}$. Recall that, since our sample is assumed to be i.i.d.,

$$\mathcal{L}_{\vec{\boldsymbol{Y}}}(\mu, \sigma^2) = \prod_{i=1}^{n} f_{Y_i}(y_i; \mu, \sigma^2)$$

(b) Derive an expression for $\ell_{\vec{\boldsymbol{Y}}}(\mu, \sigma^2)$, the log-likelihood of the sample $\vec{\boldsymbol{Y}}$. Also compute

$$\frac{\partial}{\partial \mu} \ell_{\vec{\boldsymbol{Y}}}(\mu, \sigma^2) \qquad \text{and} \qquad \frac{\partial}{\partial \sigma^2} \ell_{\vec{\boldsymbol{Y}}}(\mu, \sigma^2)$$

(c) In the two derivatives you found in part (b), replace all instances of μ with $\widehat{\mu}_{\text{MLE}}$, all instances of σ^2 with $\widehat{\sigma^2}_{\text{MLE}}$. Set the resulting expressions equal to zero and solve for $\widehat{\mu}_{\text{MLE}}$ and $\widehat{\sigma^2}_{\text{MLE}}$.

(d) The **equivariance** property of maximum likelihood estimators is as follows: given the MLE $\widehat{\theta}_{\text{MLE}}$ for a parameter θ, the MLE of $\tau(\theta)$ [where $\tau(\cdot)$ is an appropriately-behaved function] is $\tau(\widehat{\theta}_{\text{MLE}})$. For example, the MLE of θ^3 is $(\widehat{\theta}_{\text{MLE}})^3$.

Use the equivariance property and your answer to part (c) to derive an expression for the maximum likelihood estimator of σ, the population *standard deviation*.

Problem 2

Something's gone awry with *GauchoPop*'s newest bottling machine! Specifically, the new soda dispenser doesn't fill each bottle entirely - rather, the proportion Y_i of a randomly-selected bottle that is full of soda follows the distribution with density

$$f(y;\theta) = \theta y^{\theta-1} \cdot \mathbb{1}_{\{0 \leq y \leq 1\}}$$

where $\theta > 0$ is an unknown parameter. Let $Y_1, \cdots, Y_n$ denote the proportion of fill contained in n randomly-selected *GauchoPop* bottles.

(a) Find $\widehat{\theta}_{\text{MLE}}$, the maximum likelihood estimator for θ.

(b) Company regulations requires that any bottles with fewer than 0.8 fill be labeled as "not fit for sale." Let τ denote the true proportion of bottles that end up labeled as "not fit for sale" - find $\widehat{\tau}_{\text{MLE}}$, the maximum likelihood estimator for τ.

Hint: Use the equivariance property

Problem 3

Let $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim}$ Unif$[\theta, 1]$ where $\theta < 1$ is an unknown parameter.

(a) Find $\widehat{\theta}_{\text{MM}}$, the method of moments estimator for θ.

(b) Show that the likelihood of the sample $\vec{\boldsymbol{Y}}$ is given by

$$\mathcal{L}_{\vec{\mathbf{Y}}}(\theta) = \left(\frac{1}{1-\theta} \right)^n \cdot \mathbb{1}_{\{\theta \leq Y_{(1)}\}} \cdot \mathbb{1}_{\{Y_{(n)} \leq 1\}}$$

where $Y_{(1)}$ denotes the first order statistic (sample minimum) and $Y_{(n)}$ denotes the n^{th} order statistic (sample maximum). Justify your logic.

(c) Find $\widehat{\theta}_{\text{MLE}}$, the maximum likelihood estimator for θ.

(d) Find the exact sampling distribution of $\widehat{\theta}_{\text{MLE}}$, and use this to determine whether or not $\widehat{\theta}_{\text{MLE}}$ is an unbiased estimator for θ.

(e) Show that $U := Y_{(1)}$, the first order statistic, is a sufficient sufficient statistic for θ.