DISCUSSION WORKSHEET 07

PSTAT 120B: Mathematical Statistics, I
Summer Session A, 2024 with Instructor: Ethan P. Marzban

Conceptual Review

(a) What is the **Cramér-Rao Lower Bound**? How does that relate to the **Fisher Information**?

(b) What is an **efficient estimator**?

(c) What is a **confidence interval**? How do we construct one assuming normality in the population?

(d) What is a **pivotal quantity**? How do we use these to construct confidence intervals?

Problem 1:

Let $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim}$ Unif$[0, \theta]$, where both $\theta > 0$ is an unknown parameter.

(a) Propose a pivot for θ that is a function of $Y_{(n)}$, and show that your proposed quantity is in fact a pivot.

Solution: We have seen several times that

$$f_{Y_{(n)}}(y) = \frac{ny^{n-1}}{\theta^n} \cdot \mathbb{1}_{\{0 \leq y \leq \theta\}}$$

Hence, it seems plausible that $U := Y_{(1)}/\theta$ might be a pivotal quantity for θ. Let's verify this! That is, given $Y_{(n)}$ with density as stated above, we want to find the distribution of $f_U(u)$ (which we hope doesn't depend on θ).

I'll demonstrate using the Change of Variable Formula: take $g(y) = y/\theta$ so that $g^{-1}(u) = u\theta$, meaning

$$\left| \frac{\mathrm{d}}{\mathrm{d}u} g^{-1}(u) \right| = |\theta| = \theta$$

where, since $\theta > 0$, we have dropped the absolute values in the last step. Thus, by the change of variable formula,

$$\begin{aligned} f_U(u) &= f_{Y_{(n)}}(g^{-1}(u)) \cdot \left| \frac{\mathrm{d}}{\mathrm{d}u} g^{-1}(u) \right| \\ &= \frac{n(\theta u)^{n-1}}{\theta^{n-1}} \cdot \mathbb{1}_{\{0 \leq u\theta \leq \theta\}} = nu^{n-1} \cdot \mathbb{1}_{\{0 \leq u \leq 1\}} \end{aligned}$$

As this clearly doesn't depend on θ, we have shown that $U := Y_{(n)}/\theta$ is a pivotal quantity for θ.

(b) Use your pivot from part (a) to construct a $(1-\alpha) \times 100\%$ confidence interval for θ.

Solution: Following the steps to construct a confidence interval using the pivotal method, we start with a CI of the form

$$\left\{ a \leq \frac{Y_{(n)}}{\theta} \leq b \right\}$$

This is equivalent to asserting

$$\mathbb{P}\left(\frac{Y_{(n)}}{\theta} \leq a \right) = \frac{\alpha}{2} \qquad \text{and} \qquad \mathbb{P}\left(\frac{Y_{(n)}}{\theta} \geq b \right) = \frac{\alpha}{2}$$

Using the result of part (a), we find

$$\mathbb{P}\left(\frac{Y_{(n)}}{\theta} \leq a \right) = \int_0^a f_U(u)\, \mathrm{d}u = \int_0^a nu^{n-1}\, \mathrm{d}u = a^n$$
$$\mathbb{P}\left(\frac{Y_{(n)}}{\theta} \geq b \right) = \int_b^1 f_U(u)\, \mathrm{d}u = \int_b^1 nu^{n-1}\, \mathrm{d}u = 1 - b^n$$

So,

$$a^n = \frac{\alpha}{2} \implies a = \left(\frac{\alpha}{2} \right)^{1/n}; \qquad 1 - b^n = \frac{\alpha}{2} \implies b = \left(1 - \frac{\alpha}{2} \right)^{1/n}$$

Finally, we invert our interval:

$$\left\{ a \leq \frac{Y_{(n)}}{\theta} \leq b \right\} = \left\{ \frac{Y_{(n)}}{b} \leq \theta \leq \frac{Y_{(n)}}{a} \right\}$$

meaning, plugging in our values for a and b that we found before, our $(1-\alpha) \times 100\%$ CI for θ is given by

$$\left[\frac{Y_{(n)}}{\left(1 - \frac{\alpha}{2}\right)^{1/n}}, \frac{Y_{(n)}}{\left(\frac{\alpha}{2}\right)^{1/n}} \right]$$

(c) Suppose we collect the following observed instance of a sample: $\vec{\boldsymbol{y}} = (0.74, 1.97, 0.31, 0.18, 0.28)$. Construct a 95% confidence interval for θ.

Solution: We need only to plug into our CI from part (b)! Here, $y_{(n)} = 1.97$. Additionally, if we want a 95% coverage probability we should take $\alpha = 0.05$ meaning our interval is

$$\left[\frac{1.97}{(0.975)^{1/5}}, \frac{1.97}{(0.025)^{1/5}} \right] = [1.98\, ,\ 4.12]$$