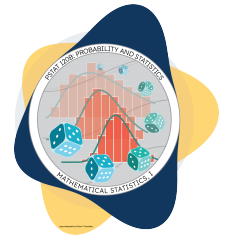


## DISCUSSION WORKSHEET 07

PSTAT 120B: Mathematical Statistics, I  
Summer Session A, 2024 with Instructor: Ethan P. Marzban



### Conceptual Review

- What is the **Cramér-Rao Lower Bound**? How does that relate to the **Fisher Information**?
- What is an **efficient estimator**?
- What is a **confidence interval**? How do we construct one assuming normality in the population?
- What is a **pivotal quantity**? How do we use these to construct confidence intervals?

### Problem 1:

Let  $Y_1, \dots, Y_n \stackrel{\text{i.i.d.}}{\sim} \text{Unif}[0, \theta]$ , where both  $\theta > 0$  is an unknown parameter.

- Propose a pivot for  $\theta$  that is a function of  $Y_{(n)}$ , and show that your proposed quantity is in fact a pivot.

**Solution:** We have seen several times that

$$f_{Y_{(n)}}(y) = \frac{ny^{n-1}}{\theta^n} \cdot \mathbb{1}_{\{0 \leq y \leq \theta\}}$$

Hence, it seems plausible that  $U := Y_{(1)}/\theta$  might be a pivotal quantity for  $\theta$ . Let's verify this! That is, given  $Y_{(n)}$  with density as stated above, we want to find the distribution of  $f_U(u)$  (which we hope doesn't depend on  $\theta$ ).

I'll demonstrate using the Change of Variable Formula: take  $g(y) = y/\theta$  so that  $g^{-1}(u) = u\theta$ , meaning

$$\left| \frac{d}{du} g^{-1}(u) \right| = |\theta| = \theta$$

where, since  $\theta > 0$ , we have dropped the absolute values in the last step. Thus, by the change of variable formula,

$$\begin{aligned} f_U(u) &= f_{Y_{(n)}}(g^{-1}(u)) \cdot \left| \frac{d}{du} g^{-1}(u) \right| \\ &= \frac{n(\theta u)^{n-1}}{\theta^{n-1}} \cdot \mathbb{1}_{\{0 \leq u\theta \leq \theta\}} = nu^{n-1} \cdot \mathbb{1}_{\{0 \leq u \leq 1\}} \end{aligned}$$

As this clearly doesn't depend on  $\theta$ , we have shown that  $U := Y_{(n)}/\theta$  is a pivotal quantity for  $\theta$ .

- (b) Use your pivot from part (a) to construct a  $(1 - \alpha) \times 100\%$  confidence interval for  $\theta$ .

**Solution:** Following the steps to construct a confidence interval using the pivotal method, we start with a CI of the form

$$\left\{ a \leq \frac{Y_{(n)}}{\theta} \leq b \right\}$$

This is equivalent to asserting

$$\mathbb{P}\left(\frac{Y_{(n)}}{\theta} \leq a\right) = \frac{\alpha}{2} \quad \text{and} \quad \mathbb{P}\left(\frac{Y_{(n)}}{\theta} \geq b\right) = \frac{\alpha}{2}$$

Using the result of part (a), we find

$$\begin{aligned} \mathbb{P}\left(\frac{Y_{(n)}}{\theta} \leq a\right) &= \int_0^a f_U(u) \, du = \int_0^a nu^{n-1} \, du = a^n \\ \mathbb{P}\left(\frac{Y_{(n)}}{\theta} \geq b\right) &= \int_b^1 f_U(u) \, du = \int_b^1 nu^{n-1} \, du = 1 - b^n \end{aligned}$$

So,

$$a^n = \frac{\alpha}{2} \implies a = \left(\frac{\alpha}{2}\right)^{1/n}; \quad 1 - b^n = \frac{\alpha}{2} \implies b = \left(1 - \frac{\alpha}{2}\right)^{1/n}$$

Finally, we invert our interval:

$$\left\{ a \leq \frac{Y_{(n)}}{\theta} \leq b \right\} = \left\{ \frac{Y_{(n)}}{b} \leq \theta \leq \frac{Y_{(n)}}{a} \right\}$$

meaning, plugging in our values for  $a$  and  $b$  that we found before, our  $(1 - \alpha) \times 100\%$  CI for  $\theta$  is given by

$$\left[ \frac{Y_{(n)}}{\left(1 - \frac{\alpha}{2}\right)^{1/n}}, \frac{Y_{(n)}}{\left(\frac{\alpha}{2}\right)^{1/n}} \right]$$

- (c) Suppose we collect the following observed instance of a sample:  $\vec{y} = (0.74, 1.97, 0.31, 0.18, 0.28)$ . Construct a 95% confidence interval for  $\theta$ .

**Solution:** We need only to plug into our CI from part (b)! Here,  $y_{(n)} = 1.97$ . Additionally, if we want a 95% coverage probability we should take  $\alpha = 0.05$  meaning our interval is

$$\left[ \frac{1.97}{(0.975)^{1/5}}, \frac{1.97}{(0.025)^{1/5}} \right] = [1.98, 4.12]$$